

# On mirabolic D-modules

MICHAEL FINKELBERG AND VICTOR GINZBURG

*To the memory of Israel Moiseevich Gelfand*

**ABSTRACT.** Let an algebraic group  $G$  act on  $X$ , a connected algebraic manifold, with finitely many orbits. For any Harish-Chandra pair  $(\mathcal{D}, G)$  where  $\mathcal{D}$  is a sheaf of twisted differential operators on  $X$ , we form a left ideal  $\mathcal{D}\mathfrak{g} \subset \mathcal{D}$  generated by the Lie algebra  $\mathfrak{g} = \text{Lie } G$ . Then,  $\mathcal{D}/\mathcal{D}\mathfrak{g}$  is a holonomic  $\mathcal{D}$ -module, and its restriction to a unique Zariski open dense  $G$ -orbit in  $X$  is a  $G$ -equivariant local system. We prove a criterion saying that the  $\mathcal{D}$ -module  $\mathcal{D}/\mathcal{D}\mathfrak{g}$  is isomorphic, under certain (quite restrictive) conditions, to a direct image of that local system to  $X$ . We apply this criterion in the special case of the group  $G = SL_n$  acting diagonally on  $X = \mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1}$ , where  $\mathcal{B}$  denotes the flag manifold for  $SL_n$ .

We further relate  $\mathcal{D}$ -modules on  $\mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1}$  to  $\mathcal{D}$ -modules on the cartesian product  $SL_n \times \mathbb{P}^{n-1}$  via a pair  $(\text{CH}, \text{HC})$ , of adjoint functors analogous to those used in Lusztig's theory of character sheaves. A second important result of the paper provides an explicit description of these functors, showing that the functor  $\text{HC}$  gives an *exact* functor on the abelian category of *mirabolic*  $\mathcal{D}$ -modules.

## Table of Contents

1. Introduction
2. A result on holonomic  $D$ -modules
3. A triple flag variety
4. Mirabolic  $D$ -modules
5. Mirabolic Harish-Chandra  $D$ -module
6. Further properties of the mirabolic Harish-Chandra  $D$ -module

## 1. INTRODUCTION

1.1. Let  $X$  be a smooth connected algebraic variety and fix  $(\mathcal{D}, G)$ , a Harish-Chandra pair on  $X$  in the sense of [BB3], §1.8.3. Thus,  $G$  is an algebraic group acting on  $X$ , and  $\mathcal{D}$  is a sheaf of (twisted) differential operators on  $X$ . One also has a Lie algebra map  $\mathfrak{g} := \text{Lie } G \rightarrow \mathcal{D}$  and  $\mathcal{D}\mathfrak{g} \subset \mathcal{D}$ , the left ideal generated by the image of this map.

Assume now that  $\dim X = \dim G$  and that  $G$  is a reductive group acting on  $X$  with finitely many orbits. Let  $U$  be a unique Zariski open dense  $G$ -orbit in  $X$ . Then, the quotient  $\mathcal{V} := \mathcal{D}/\mathcal{D}\mathfrak{g}$  is a holonomic  $\mathcal{D}$ -module on  $X$ , and  $\mathcal{V}^\circ := j^*\mathcal{V}$ , the restriction of  $\mathcal{V}$  via the open imbedding  $j : U \hookrightarrow X$ , is a  $G$ -equivariant (twisted) local system on  $U$ . The main result of section 2 (Theorem 2.3.1) says that the  $\mathcal{D}$ -module  $\mathcal{D}/\mathcal{D}\mathfrak{g}$  is isomorphic, under certain conditions on the moment map  $T^*X \rightarrow \mathfrak{g}^*$  and certain bounds on the roots of an associated  $b$ -function, cf. Remark 5.5.2, to either  $j_*\mathcal{V}^\circ$  or  $j_!\mathcal{V}^\circ$ , a direct image of the local system  $\mathcal{V}^\circ$  to  $X$ .

We are mostly interested in the special case where  $G = SL_n$ , and  $X = \mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1}$ . Here,  $\mathcal{B}$  denotes the flag variety of the group  $SL_n$ , so  $X$  is a sort of ‘triple flag manifold’. It turns out that the group  $G$  acts diagonally in  $X$  with *finitely many* orbits (all cases of such an ‘unusual’ phenomena have been classified in [MWZ]). Moreover, the theorem from §2 applies in this case.

Our interest in  $\mathcal{D}$ -modules on  $X = \mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1}$  is due to their close connection with *mirabolic character  $\mathcal{D}$ -modules*.<sup>1</sup> These are certain  $\mathcal{D}$ -modules on the space  $G \times \mathbb{P}^{n-1}$ , introduced in [GG],

<sup>1</sup>The name ‘mirabolic’ comes from the combination ‘miraculous parabolic’, which is the parabolic subgroup  $P \subset G$  that fixes a point in  $\mathbb{P}^{n-1}$ . This parabolic was first considered in the work by Gelfand and Kazhdan [GK], and it has some very specific features. In our situation, considering  $G$ -equivariant  $\mathcal{D}$ -modules on the space  $G \times \mathbb{P}^{n-1}$  is equivalent, essentially, to considering  $P$ -equivariant  $\mathcal{D}$ -modules on  $G$ . For this reason, we use the name ‘mirabolic’.

and studied further in [FG]. The category of mirabolic  $\mathcal{D}$ -modules plays an important role in the study of category  $\mathcal{O}$  for a rational Cherednik algebra, cf. [GG], [FG], and references therein.

One of our key observations is that (a derived version of) the category of mirabolic  $\mathcal{D}$ -modules is related to a (derived) category of  $\mathcal{D}$ -modules on the space  $\mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1}$  via a pair (CH, HC) of adjoint functors. Here, the *character functor* CH is a ‘mirabolic counterpart’ of a similar functor used by Lusztig [Lu] in his theory of *character sheaves* on the group  $G$ . The functor HC is a ‘mirabolic counterpart’ of the *Harish-Chandra* functor considered in [G2].

The second important result of the paper is a mirabolic analogue of a result from [BFO]. Specifically, part (ii) of our Theorem 4.4.4 says that the functor HC gives an exact functor from the *abelian* category of mirabolic  $\mathcal{D}$ -modules to an appropriately defined abelian category of Harish-Chandra modules.

There is one especially important mirabolic  $\mathcal{D}$ -module, called *Harish-Chandra  $\mathcal{D}$ -module*. To define it, write  $\mathcal{D}^c$  for the sheaf of twisted differential operators on  $G \times \mathbb{P}^{n-1}$  with twist  $c \in \mathbb{C}$  along the second factor. Let  $\Delta : \mathrm{Lie} G \rightarrow \mathcal{D}^c$  be the map induced by the differential of the  $G$ -diagonal action on  $G \times \mathbb{P}^{n-1}$ , via  $g : (g', l) \mapsto (gg'g^{-1}, g(l))$ , and put  $\mathfrak{g} := \Delta(\mathrm{Lie} G) \subset \mathcal{D}^c$ . Further, let  $\mathcal{D}(G)^{G \times G}$  be the algebra of  $G$ -bi-invariant differential operators on  $G$ . Thus  $\mathcal{D}(G)^{G \times G}$  is a commutative algebra isomorphic to the center of  $\mathcal{U}(\mathrm{Lie} G)$ , the universal enveloping algebra of the Lie algebra  $\mathrm{Lie} G$ .

Following [GG], §7.4, for any maximal ideal  $\mathfrak{z}_\theta \subset \mathcal{D}(G)^{G \times G}$ , one defines an associated *mirabolic Harish-Chandra  $\mathcal{D}$ -module* to be, cf. Definition 5.1.1 for more details,

$$\mathcal{G}^{\theta, c} := \mathcal{D}^c / (\mathcal{D}^c \mathfrak{g} + \mathcal{D}^c (\mathfrak{z}_\theta \otimes 1)). \quad (1.1.1)$$

This is a  $G$ -equivariant, holonomic  $\mathcal{D}$ -module on  $G \times \mathbb{P}^{n-1}$ , analogous to a  $\mathcal{D}$ -module on the group  $G$  itself studied by Hotta-Kashiwara in [HK1], [HK2]. In the last section of the paper, we use the functor CH and our general result from section 2 to obtain a purely geometric construction of the perverse sheaf that corresponds to the mirabolic Harish-Chandra  $\mathcal{D}$ -module  $\mathcal{G}^{\theta, c}$  and give a conjectural description of the restriction of  $\mathcal{G}^{\theta, c}$  to an open dense subset of  $G \times \mathbb{P}^{n-1}$  (see Conjecture 6.4.1).

**1.2. Acknowledgements.** We are very much indebted to Roman Bezrukavnikov for communicating to us his ideas, related to the results of [BFO], before they were made public. We are grateful to Sasha Beilinson for his kind explanations concerning monodromic  $\mathcal{D}$ -modules. We also thank an anonymous referee for his helpful comments. M.F. is grateful to IAS, to the University of Chicago, and to Indiana University at Bloomington for the hospitality and support. The work of M.F. was partially supported by the RFBR grant 09-01-00242 and the Science Foundation of the SU-HSE awards No.09-08-0008 and No.09-09-0009. The work of V.G. was partially supported by the NSF grant DMS-0601050.

## 2. A RESULT ON HOLONOMIC $\mathcal{D}$ -MODULES

**2.1.** For any smooth variety  $X$ , write  $\Omega_X^p$  for the sheaf of algebraic differential  $p$ -forms on  $X$ , and  $\omega_X$  for the canonical line bundle of top degree differential forms. Further, let  $(\Omega_X^p)_{\mathrm{closed}} \subset \Omega_X^p$  denote the subsheaf of closed  $p$ -forms. Abusing the notation, we write  $\omega \in H^2(X, (\Omega_X^1)_{\mathrm{closed}})$  for the Chern class of the canonical bundle  $\omega_X$ .

We refer the reader to [BB3] for generalities on twisted differential operators (TDO), and to [K2], §2, for a survey of basic functors on twisted  $\mathcal{D}$ -modules.

We say that  $\mathcal{D}$ , a sheaf of TDO on  $X$ , is *locally trivial* if, locally in étale topology, one has an isomorphism  $\mathcal{D} \cong \mathcal{D}_X$ , where  $\mathcal{D}_X$  stands for the sheaf of (nontwisted) algebraic differential operators on  $X$ . It is known that the sheaves of algebraic locally trivial TDO are parametrized (up to isomorphism) by elements of the group  $H_{\mathrm{ét}}^1(X, (\Omega_X^1)_{\mathrm{closed}})$ , [BB3].

All TDO considered in this paper are assumed, without further notice, to be locally trivial. Given  $\mathcal{D}_\chi$ , the sheaf of twisted differential operators on  $X$  associated with a class  $\chi \in H^2(X, (\Omega_X^1)_{\text{closed}})$ , we write  $\mathcal{D}_{-\chi}$  for the sheaf of differential operators with the opposite twisting.

Given an associative algebra, resp. sheaf of algebras,  $A$ , write  $A^{\text{op}}$  for the opposite algebra. Thus, for any TDO  $\mathcal{D}_\chi$ , one has the sheaf  $\mathcal{D}_\chi^{\text{op}} := (\mathcal{D}_\chi)^{\text{op}}$ . There is a canonical isomorphism of TDO's  $\mathcal{D}_\chi^{\text{op}} \cong \mathcal{D}_{\omega-\chi}$ , see [K2], 2.7.

We will use the notion of *regular singularities* for modules over (locally trivial) TDO. Let  $\mathcal{D}$  be such a TDO on  $X$  and let  $j : U \hookrightarrow X$  be an open imbedding. Let  $M$  be a holonomic  $j^*\mathcal{D}$ -module. Locally in étale topology, one may view  $j_*M$  as a module over  $\mathcal{D}_X$ , the sheaf of nontwisted differential operators on  $X$ . We say that  $M$  is *regular at a point*  $x \in X$  if the following holds: *For any smooth curve  $C \subset X$  containing  $x$ , the restriction of  $j_*M$  to  $C$  is either supported at the point  $x$ , or is smooth at  $x$ , or else it has a regular singularity at  $x$ .*

2.2. Let a connected algebraic group  $G$  act on  $X$ , a smooth connected variety. Let  $U \subset X$  be a (unique) Zariski open dense  $G$ -orbit, and assume  $X \setminus U$  is a hypersurface, not necessarily irreducible, in general.

Let  $\mathcal{L}$  be a  $G$ -linearization of the line bundle  $\mathcal{O}_X(Y)$  where  $Y$  is a divisor with support equal to that hypersurface. Thus,  $\mathcal{L}$  is a  $G$ -equivariant line bundle on  $X$  and there exists a regular section  $s$ , of  $\mathcal{L}$ , such that  $U = X \setminus s^{-1}(0)$ . The following result is well known.

**Lemma 2.2.1.** *There exists a group homomorphism  $\phi : G \rightarrow \mathbb{C}^\times$  and a  $\phi$ -semi-invariant regular section  $s \in \Gamma(X, \mathcal{L})$ , such that*

$$U = X \setminus s^{-1}(0) \quad \text{and, we have} \quad g^*(s) = \phi(g) \cdot s, \quad \forall g \in G. \quad (2.2.2)$$

*Proof.* Let  $s \in \Gamma(X, \mathcal{L})$  be any regular section such that  $U = X \setminus s^{-1}(0)$ . Then, for any  $g \in G$ , we have  $(g^*s) \cdot s^{-1} \in \mathcal{O}(X)^\times$ , is an invertible regular function on  $X$ . The assignment  $g \mapsto (g^*s) \cdot s^{-1}$  gives an algebraic cocycle that defines a cohomology class in  $H_{\text{alg}}^1(G, \mathcal{O}(X)^\times)$ .

Now, since  $G$  is connected, from [KKV], Proposition 5.1, we deduce that any cohomology class in  $H_{\text{alg}}^1(G, \mathcal{O}(X)^\times)$ , may be represented by the class of a character  $\phi : G \rightarrow \mathbb{C}^\times$ . Therefore, there exists a character  $\phi$  and an invertible function  $f \in \mathcal{O}(X)^\times$  such that we have  $(g^*s) \cdot s^{-1} = \phi(g) \cdot (g^*f) \cdot f^{-1}$ , for any  $g \in G$ . It follows that  $f^{-1} \cdot s$  is a  $\phi$ -semi-invariant section with the required properties.  $\square$

For any  $x \in X$ , we let  $G_x \subset G$  denote the isotropy group of the point  $x$ . Write  $\mathfrak{g} := \text{Lie } G$ , resp.  $\mathfrak{g}_x := \text{Lie } G_x$ ,  $\forall x \in X$ . The  $G$ -equivariant structure on  $\mathcal{L}$  gives rise, for any  $x \in X$ , to a group homomorphism  $\chi_{\mathcal{L},x} : G_x \rightarrow \mathbb{C}^\times$ , induced by the  $G_x$ -action in the fiber of  $\mathcal{L}$  at  $x$ . Observe that, for any  $x \in U$ , we have  $\phi|_{G_x} = \chi_{\mathcal{L},x}$ . We keep the same notation,  $\phi$  and  $\chi_{\mathcal{L},x}$ , for Lie algebra homomorphisms corresponding to the group characters introduced above.

2.3. Let  $(\mathcal{D}, G)$  be a Harish-Chandra algebra on  $X$ , in the sense of [BB3], §1.8.3; Put  $\mathfrak{g} := \text{Lie } G$ , and let  $\mathfrak{g} \rightarrow \mathcal{D}$ ,  $u \mapsto \vec{u}$  be the corresponding Lie algebra homomorphism.

Given a  $G$ -equivariant line bundle  $\mathcal{L}$  on  $X$  we introduce the following notation

$$\mathcal{D}_k := \mathcal{L}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes(-k)}, \quad \mathcal{F}(k) := \mathcal{L}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{F}, \quad k \in \mathbb{Z},$$

for any  $\mathcal{O}_X$ -module  $\mathcal{F}$ . It is clear that, for any  $k \in \mathbb{Z}$ , the pair  $(\mathcal{D}_k, G)$  has a structure of Harish-Chandra algebra as well, in particular, there is a Lie algebra map  $\mathfrak{g} \rightarrow \mathcal{D}_k$ ,  $u \mapsto \vec{u}$ .

One also has  $(\mathcal{D}_k)^{\text{op}}$ , an opposite TDO, and there is a natural isomorphism  $(\mathcal{D}_k)^{\text{op}} \cong (\mathcal{D}^{\text{op}})_{-k}$ . From now on, we will write  $\mathcal{D}_{-k}^{\text{op}} := (\mathcal{D}^{\text{op}})_{-k}$ . For any  $\mathcal{D}$ -module  $\mathcal{F}$ , the sheaf  $\mathcal{F}(k)$  has a natural  $\mathcal{D}_k$ -module structure. Thus, a right  $\mathcal{D}_k$ -module is the same thing as a left  $(\mathcal{D}_{-k}^{\text{op}})$ -module.

Given a 1-dimensional character  $\psi : \mathfrak{g} \rightarrow \mathbb{C}$ , let  $\mathfrak{g}^\psi \subset \mathcal{D}$  denote the image of  $\mathfrak{g}$  under the Lie algebra morphism  $u \mapsto \vec{u} - \psi(u) \cdot 1$ . Let  $\mathcal{D}\mathfrak{g}^\psi \subset \mathcal{D}$  be the left ideal generated by  $\mathfrak{g}^\psi$ , and let  $\mathcal{D}/\mathcal{D}\mathfrak{g}^\psi$  be the corresponding left  $\mathcal{D}$ -module on  $X$ .

Let  $U \subset X$  be a Zariski open dense  $G$ -orbit and write  $j : U \hookrightarrow X$  for the imbedding. Fix  $k \in \mathbb{Z}$ , and let  $j^*(\mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^\psi)$  be the restriction of the  $\mathcal{D}_k$ -module  $\mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^\psi$  to  $U$ . It is clear that the  $G$ -action on  $\mathcal{D}_k$  makes  $j^*(\mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^\psi)$  a  $G$ -equivariant locally free coherent  $\mathcal{O}_U$ -module. Furthermore, if  $\psi|_{\mathfrak{g}_x} \neq 0$  for some  $x \in U$  then, one must have  $j^*(\mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^\psi) = 0$ . Otherwise,  $j^*(\mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^\psi)$  is a line bundle on  $U$ .

Let  $\delta : \mathfrak{g} \rightarrow \mathbb{C}$ ,  $u \mapsto \text{Tr ad } u$  be the *modular character* of the Lie algebra  $\mathfrak{g}$ .

**Theorem 2.3.1.** *Let a connected reductive group  $G$  act on a smooth variety  $X$ , such that  $\dim X = \dim G$ , with finitely many orbits. Let  $(\mathcal{D}, G)$  be a Harish-Chandra pair, where  $\mathcal{D}$  is a locally trivial TDO on  $X$ . Then, for any character  $\psi : \mathfrak{g} \rightarrow \mathbb{C}$  and any  $k \in \mathbb{Z}$ , one has*

(i) *The  $\mathcal{D}_k$ -module  $\mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^\psi$  is holonomic and regular at any point  $x \in X$ , cf. §2.1; furthermore, there is a natural isomorphism*

$$R\mathcal{H}om_{\mathcal{D}_k}(\mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^\psi, \mathcal{D}_k) \cong \mathcal{D}_{-k}^{\text{op}}/\mathcal{D}_{-k}^{\text{op}}\mathfrak{g}^{\delta-\psi}[-\dim X].$$

(ii) *Let  $U \subset X$  be a Zariski open dense  $G$ -orbit in  $X$ , let  $\phi : G \rightarrow \mathbb{C}^\times$  be as in Lemma (2.2.1) and assume, in addition, that*

$$\chi_{\mathcal{L},x} \neq \phi|_{\mathfrak{g}_x}, \quad \forall x \in X \setminus U. \quad (2.3.2)$$

*Then, one has canonical isomorphisms:*

$$\begin{aligned} \mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^{\psi+k\cdot\phi} &\xrightarrow{\sim} j_*j^*(\mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^{\psi+k\cdot\phi}), & \text{for } k \leq 0; \\ j!j^*(\mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^{\psi+k\cdot\phi}) &\xrightarrow{\sim} \mathcal{D}_k/\mathcal{D}_k\mathfrak{g}^{\psi+k\cdot\phi}, & \text{for } k \geq 0. \end{aligned} \quad (2.3.3)$$

It would be interesting to apply this theorem to spherical varieties, and also to some prehomogeneous vector spaces.

The rest of this section is devoted to the proof of Theorem 2.3.1.

**2.4. Geometry of the moment map.** In this section, we prove a few geometric results which will enable us to reduce the proof of Theorem 2.3.1 to Proposition 2.5.2 below.

Given an arbitrary smooth  $G$ -variety  $X$  and a  $G$ -equivariant line bundle  $\mathcal{L}$ , on  $X$ , write  $\mathcal{L}^*$  for the dual line bundle. Let  $L \rightarrow X$  be a principal  $\mathbb{C}^\times$ -bundle obtained by removing the zero section in the total space of the line bundle  $\mathcal{L}^*$ . The  $\mathbb{C}^\times$ -action on  $L$  commutes with the natural  $G$ -action, hence makes  $L$  a  $G \times \mathbb{C}^\times$ -variety.

Let  $T^*L$  be the total space of the cotangent bundle on  $L$ . One has a natural Hamiltonian  $G \times \mathbb{C}^\times$ -action on  $T^*L$ , with moment map  $T^*L \rightarrow \mathfrak{g}^* \times (\text{Lie } \mathbb{C}^\times)^*$ . Observe that the action of the second factor  $\mathbb{C}^\times \subset G \times \mathbb{C}^\times$  on  $T^*L$  is free. Thus, the quotient  $Z := (T^*L)/\mathbb{C}^\times$  is a smooth variety, and the moment map above descends to the quotient. The resulting map may be written in the form of a cartesian product of two maps  $\mu \times \nu : Z \rightarrow \mathfrak{g}^* \times \mathbb{C}$ .

It is well known that the map  $\nu : Z \rightarrow \mathbb{C}$  is a smooth morphism, and for each  $a \in \mathbb{C}$ , the fiber  $\nu^{-1}(a)$  is a symplectic manifold. Furthermore, for  $a = 0$ , one has a canonical isomorphism  $\nu^{-1}(0) \cong T^*X$ , such that the restriction of the map  $\mu$  to  $\nu^{-1}(0)$  may be identified with

$$\mu = \mu|_{\nu^{-1}(0)} : \nu^{-1}(0) = T^*X \rightarrow \mathfrak{g}^*,$$

the moment map for the natural Hamiltonian  $G$ -action on  $T^*X$ .

There is also a (non-Hamiltonian)  $\mathbb{C}^\times$ -action on  $T^*L$ , by dilations along the fibers. This action, to be referred to as ‘dot-action’, descends to a  $\mathbb{C}^\times$ -action  $\mathbb{C}^\times \ni a : z \mapsto a \cdot z$ , on  $Z = (T^*L)/\mathbb{C}^\times$ , the quotient of  $T^*L$  by the Hamiltonian  $\mathbb{C}^\times$ -action. There is also a ‘dot-action’ on  $\mathfrak{g}^* \times \mathbb{C}$ , defined as the

$\mathbb{C}^\times$ -diagonal action by dilations. The dot-actions of  $\mathbb{C}^\times$  on  $Z$  and on  $\mathfrak{g}^* \times \mathbb{C}$  are both *contractions*, and the map  $\mu \times \nu : Z \rightarrow \mathfrak{g}^* \times \mathbb{C}$  is clearly dot-equivariant.

Let  $s \in \Gamma(X, \mathcal{L})$  be a  $\phi$ -semi-invariant section, so that (2.2.2) holds. Put  $U := X \setminus s^{-1}(0)$ , and let  $\tilde{U}$  denote the preimage of  $U$  in  $L$ , the total space of  $\mathcal{L}^*$ .

The section  $s$  may (and will) be viewed as a regular function on  $L$ . The graph of the closed 1-form  $d \log s = s^{-1} \cdot ds$  may be viewed as a section of  $T^*L$  over  $\tilde{U}$ . Let  $\Lambda \subset Z$  be the image of the graph of  $d \log s$  under the projection  $T^*L \twoheadrightarrow (T^*L)/\mathbb{C}^\times = Z$ .

**Lemma 2.4.1.** *Let  $U \subset X$  be a Zariski open  $G$ -orbit such that (2.2.2) holds. Then,*

(i) *The set  $\Lambda$  is a smooth, closed Lagrangian submanifold of the symplectic manifold  $\nu^{-1}(1)$ . Furthermore, we have*

$$\Lambda \subset \mu^{-1}(\phi) \cap \nu^{-1}(1).$$

(ii) *Assume also that the following conditions (2.4.2)-(2.4.3) hold:*

• *The intersection  $\mu^{-1}(\phi) \cap \nu^{-1}(1)$  is reduced at any point of  $\Lambda$ ;* (2.4.2)

• *The group  $G$  acts on  $X$  with finitely many orbits.* (2.4.3)

*Then,  $\Lambda$  is an irreducible component of the scheme  $\mu^{-1}(\phi) \cap \nu^{-1}(1)$ .*

(iii) *If, in addition, condition (2.3.2) holds then  $\mu^{-1}(\phi) \cap \nu^{-1}(1) = \Lambda$ , is an irreducible scheme.*

*Proof.* To prove (i) we observe that the function  $s$  on  $L$  associated with any section of  $\mathcal{L}$  is a degree 1 homogeneous function, that is, we have  $s(a \cdot \ell) = a \cdot s(\ell)$ , for any  $a \in \mathbb{C}^\times$ ,  $\ell \in L$ . For such a function, the graph of  $d \log s$  is stable under the Hamiltonian  $\mathbb{C}^\times$ -action on  $T^*L$  and is contained in the fiber of the moment map over the subset  $\mathfrak{g}^* \times \{1\} \subset \mathfrak{g}^* \times \mathbb{C}$ . Trivializing the line bundle  $\mathcal{L}$  locally, one sees (cf. [K2, §5]) that this graph is a smooth and closed subvariety of  $T^*L$ . It follows that  $\Lambda$ , the quotient of the graph by a free  $\mathbb{C}^\times$ -action, is a smooth and closed subvariety of  $\nu^{-1}(1)$ .

Observe further, for a  $\phi$ -semi-invariant section of  $\mathcal{L}$ , one has  $\vec{u}(s) = \phi(u) \cdot s$ ,  $\forall u \in \mathfrak{g}$ . This equation yields  $\Lambda \subset \mu^{-1}(\phi)$ , and part (i) follows.

To prove (ii), let  $S$  denote the *finite* set of all  $G$ -orbits in  $X$ . It is well known that, set-theoretically, we have that  $\mu^{-1}(0) = \bigsqcup_{O \in S} T_O^*X$ , is the union of the conormal bundles to  $G$ -orbits. Thus, we compute

$$\dim(\mu^{-1}(0) \cap \nu^{-1}(0)) = \dim \mu^{-1}(0) = \dim \left( \bigsqcup_{O \in S} T_O^*X \right) = \frac{1}{2} \cdot \dim T^*X = \dim X. \quad (2.4.4)$$

Moreover, the above shows that the dimension of *any* irreducible component of the intersection  $\mu^{-1}(0) \cap \nu^{-1}(0)$  equals  $\dim X$ . Further, the dot-action being a contraction, we deduce an inequality  $\dim[\mu^{-1}(\phi) \cap \nu^{-1}(1)] \leq \dim[\mu^{-1}(0) \cap \nu^{-1}(0)]$ . Thus, using part (i), we obtain

$$\dim \Lambda \leq \dim[\mu^{-1}(\phi) \cap \nu^{-1}(1)] \leq \dim[\mu^{-1}(0) \cap \nu^{-1}(0)] = \dim X = \dim \Lambda.$$

Thus, we have  $\dim \Lambda = \dim[\mu^{-1}(1) \cap \nu^{-1}(1)]$ , and part (ii) of the lemma follows from (2.4.2).

To prove (iii), consider the following diagram

$$\begin{array}{ccc} \mu^{-1}(\mathbb{C} \cdot \phi) \cap \nu^{-1}(1) & \hookrightarrow & \nu^{-1}(1) \\ \mu \downarrow & & \mu \downarrow \\ \mathbb{C} \cdot \phi & \hookrightarrow & \mathfrak{g}^* \end{array} \quad \begin{array}{c} \searrow \text{pr}_X \\ \\ X \supset U. \end{array} \quad (2.4.5)$$

Part (ii) of the lemma implies that we have  $\text{pr}_X^{-1}(U) \cap \mu^{-1}(\phi) \cap \nu^{-1}(1) = \Lambda$ .

We leave to the reader to verify that, for any point  $x \in \text{pr}_X(\mu^{-1}(\phi) \cap \nu^{-1}(1))$ , one must have  $\chi_{\mathcal{L},x} = \phi|_{\mathfrak{g}_x}$ . Hence, condition (2.3.2) insures that  $\text{pr}_X(\mu^{-1}(\phi) \cap \nu^{-1}(1)) \subset U$ , and (iii) is proved.  $\square$

**Lemma 2.4.6.** *Let the group  $G$  act on  $X$  with finitely many orbits. Then, we have  $\dim \mu^{-1}(0) = \dim X$ . If, in addition,  $\dim X = \dim G$ , then the following holds:*

- (i) *Each of the two maps  $\mu : T^*X \rightarrow \mathfrak{g}^*$  and  $\mu \times \nu : Z \rightarrow \mathfrak{g}^* \times \mathbb{C}$  is flat;*
- (ii) *Any fiber of the moment map  $\mu$  is a complete intersection in  $T^*X$ ;*
- (iii) *The two conditions of Lemma 2.4.1(ii) hold.*

*Proof.* The equation  $\dim \mu^{-1}(0) = \dim X$  is clear from (2.4.4).

Observe next that each of the maps  $\mu$  and  $\mu \times \nu$ , is an equivariant morphism between smooth varieties with contracting  $\mathbb{C}^\times$ -actions, the ‘dot-actions’. In such a case, the dimension of any fiber of the morphism is less than or equal to the dimension of the zero fiber. Thus, any fiber of either  $\mu$  or  $\mu \times \nu$ , has dimension less than or equal to  $\dim X$ . Now, the assumption that  $\dim X = \dim G$  implies that the fiber dimension is equal to  $\dim T^*X - \dim \mathfrak{g}^*$ . We conclude that each of the maps  $\mu \times \nu$  and  $\mu$  is flat. This yields part (i).

Part (ii) also follows, since  $T^*X$  and  $\mathfrak{g}^*$  are smooth schemes and any fiber of a flat morphism of smooth schemes is a complete intersection.

We now prove (iii). Condition (2.4.3) is clear. To prove (2.4.2), we use diagram (2.4.5), and set  $\mu^{-1}(\mathbb{C} \cdot \phi)_U := \text{pr}_X^{-1}(U) \cap \mu^{-1}(\mathbb{C} \cdot \phi)$ . This is an open subset in  $\mu^{-1}(\mathbb{C} \cdot \phi)$ . Observe that, since the group  $G$  acts transitively on  $U$  with finite stabilizers, it follows that the scheme-theoretic intersection  $\mu^{-1}(0 \cdot \phi)_U \cap \nu^{-1}(0)$  is the *reduced* zero section  $U \subset T^*U$ . Hence, the general fiber of the map  $\nu : \mu^{-1}(\mathbb{C} \cdot \phi)_U \rightarrow \mathbb{C} \cdot \phi$  is reduced as well. But *any* nonzero fiber of this map may be viewed as ‘general’, due to the  $\mathbb{C}^\times$ -action. Thus, the fiber  $\mu^{-1}(\mathbb{C} \cdot \phi)_U \cap \nu^{-1}(1) \supset \Lambda$  is reduced, and (2.4.2) is proved.  $\square$

**2.5. Reduction of the proof of Theorem 2.3.1.** We need to review some basic definitions concerning  $G$ -monodromic  $\mathcal{D}$ -modules.

Let  $\psi : \mathfrak{g} \rightarrow \mathbb{C}$  be a 1-dimensional character. First of all, we introduce a flat connection on  $\mathcal{O}_G$ , a rank one trivial line bundle on  $G$ , defined by the formula  $\nabla_u(f) := u(f) - \psi(u) \cdot f$ , for any  $f \in \mathcal{O}_G$  and any left invariant vector field  $u$  on  $G$  identified with the corresponding element of the Lie algebra  $\mathfrak{g}$ . This connection makes the structure sheaf  $\mathcal{O}_G$  a  $\mathcal{D}_G$ -module, to be denoted  $\mathcal{O}_G^\psi$ .

Now, let  $X$  be an arbitrary smooth  $G$ -variety, with the action map  $a : G \times X \rightarrow X$ . In the setting of §2.3, let  $(\mathcal{D}, G)$  be a Harish-Chandra pair on  $X$ . By [BB3, §1.4.5(ii)], one has a natural isomorphism  $a^*\mathcal{D} \cong \mathcal{D}_G \boxtimes \mathcal{D}$ , of TDO.

Let  $M$  be a  $\mathcal{D}$ -module on  $X$ . We say that  $M$  is  $(G, \psi)$ -*monodromic* if  $M$  is a *weakly*  $G$ -equivariant  $\mathcal{D}$ -module (i.e. a ‘weak  $(\mathcal{D}, G)$ -module’ in the sense of [BB3, §1.8.5]) and there is an isomorphism  $a^*M \cong \mathcal{O}_G^\psi \boxtimes M$ , of  $\mathcal{D}_G \boxtimes \mathcal{D}$ -modules, such that the natural cocycle condition holds.

The following result is an extension of [Bo, VII, §12.11], where a similar result was proved for (strongly)  $G$ -equivariant  $\mathcal{D}$ -modules.

**Lemma 2.5.1.** *Let  $G$ , a reductive group, act on  $X$  with finitely many orbits. Let  $(\mathcal{D}, G)$  be a Harish-Chandra pair, where  $\mathcal{D}$  is a locally trivial TDO on  $X$ . Then, any  $(G, \psi)$ -monodromic  $\mathcal{D}$ -module is regular at every point  $x \in X$ .*

*Proof.* We follow the same strategy as in the proof of [HTT, Theorem 11.6.1], cf. also [Bo, VII, §12.11]. First of all, we observe that  $\mathcal{O}_G^\psi$  is a regular  $\mathcal{D}_G$ -module in the usual sense (i.e. for any completion  $j : G \hookrightarrow \overline{G}$ , the  $\mathcal{D}_{\overline{G}}$ -module  $j_*\mathcal{O}_G^\psi$  is regular at any point  $g \in \overline{G}$ ), provided the group  $G$  is reductive. This is verified directly in the case where  $G$  is a complex torus; the case of a general reductive group  $G$  can be easily reduced to the case of a torus.

Now, let  $Y$  be an arbitrary smooth  $G$ -variety and let  $(\mathcal{D}, G)$  be a Harish-Chandra pair, where  $\mathcal{D}$  is a locally trivial TDO on  $Y$  and  $G$  is a reductive group. Let  $\iota : X \hookrightarrow Y$  be an imbedding of a

$G$ -stable smooth locally closed subvariety. There is a well defined pull-back  $i^*\mathcal{D}$ , a TDO on  $X$ , cf. [BB3, §1.4].

We claim the following: *If  $X$  is a finite union of  $G$ -orbits, then any  $(G, \psi)$ -monodromic  $i^*\mathcal{D}$ -module  $M$ , on  $X$ , is regular at any point  $y \in Y$  (this statement is vacuous unless  $y$  is contained in the closure of  $X$ ). In the special case where  $X = G/K$  is a single  $G$ -orbit, the claim is a straightforward consequence of the regularity of the  $\mathcal{D}_G$ -module  $\mathcal{O}_G^\psi$ .*

We now prove the claim in the general case. Let  $O$  be a closed  $G$ -orbit in  $X$ . Write  $i : O \hookrightarrow X$  for the imbedding, and write  $a_O$ , resp.  $a_X$ , for the  $G$ -action morphism on  $O$ , resp. on  $X$ . Let  $i^!M$  be the (derived) restriction of  $M$ , a  $(G, \psi)$ -monodromic  $\mathcal{D}$ -module on  $X$ , to  $O$ . Thus,  $i^!M$  is a complex and each cohomology group  $\mathcal{H}^p(i^!M)$ , of that complex, is a holonomic  $i^*\mathcal{D}$ -module on  $O$ . Furthermore, we observe that  $\mathcal{H}^p(i^!M)$  is a  $(G, \psi)$ -monodromic  $i^*\mathcal{D}$ -module. This follows easily by equating compositions of derived restriction functors induced by the following equal composite maps  $i \circ a_O = a_X \circ (\text{Id}_G \times i) : G \times O \rightarrow X$ .

We deduce, using the result in the case of one orbit, that each cohomology group  $\mathcal{H}^p(i^!M)$  is regular at any point  $y \in Y$ . The proof of the claim is now completed by induction on the number of  $G$ -orbits in  $X$ . The argument is based on a long exact sequence (the latter works for locally trivial TDO similarly to the usual case) in the same way as in the proof of [HTT, Theorem 11.6.1].

Finally, applying the claim in the case  $X = Y$  yields the statement of the lemma.  $\square$

We now return to the setting of Theorem 2.3.1. It is immediate from definitions that  $\mathcal{D}_k/\mathcal{D}_k \mathfrak{g}^\psi$  is a  $(G, \psi)$ -monodromic  $\mathcal{D}_k$ -module. The number of  $G$ -orbits on  $X$  being finite and the group  $G$  being reductive, we deduce from Lemma 2.5.1 that  $\mathcal{D}_k/\mathcal{D}_k \mathfrak{g}^\psi$  is a holonomic  $\mathcal{D}_k$ -module which is regular at every  $x \in X$ . At this point, it is clear that Theorem 2.3.1 follows from Lemma 2.4.1 and the following more general result.

**Proposition 2.5.2.** *Let  $G$  act on  $X$  with finitely many orbits. Then,*

(i) *If  $\dim X = \dim G$  then, for any character  $\psi : \mathfrak{g} \rightarrow \mathbb{C}$ , there is a natural isomorphism*

$$R\mathcal{H}om_{\mathcal{D}_k}(\mathcal{D}_k/\mathcal{D}_k \mathfrak{g}^\psi, \mathcal{D}_k) \cong \mathcal{D}_{-k}^{\text{op}}/\mathcal{D}_{-k}^{\text{op}} \mathfrak{g}^{\delta-\psi}[-\dim X], \quad \forall k \in \mathbb{Z}.$$

(ii) *Assume that conditions (2.4.2)-(2.4.3) as well as (2.3.2) hold and, moreover, that  $\mathcal{D}$  is a locally trivial TDO. Then, for all sufficiently negative integers  $k \ll 0$ , the canonical map in (2.3.3) is an isomorphism.*

Part (i) of the proposition is an immediate consequence of Lemma 2.4.6, and of Lemma 6.1.1 of section 6.1 below. The proof of part (ii) is based on some results of Kashiwara and will be given in the following subsection.

**2.6. An application of the  $b$ -function.** We begin with the following general result whose proof is obtained by a standard application of the theory of  $b$ -functions, cf. [K1]. Let  $X$  be a manifold,  $f : X \rightarrow \mathbb{C}$  be a regular function, and  $U := X \setminus f^{-1}(0) \xrightarrow{j} X$ . Let  $\mathcal{D}$  be a locally trivial TDO on  $X$ , and write  $\mathcal{D}_U$  for the restriction of  $\mathcal{D}$  to the open set  $U$ .

**Lemma 2.6.1.** *Let  $\mathcal{E} = \mathcal{D}_U \cdot e$  be a cyclic, holonomic  $\mathcal{D}_U$ -module generated by an element  $e \in \mathcal{E}$ . Then, for any  $k \ll 0$ , the element  $f^k \cdot e$  is a generator for the  $\mathcal{D}$ -module  $j_*\mathcal{E}$ , that is, we have  $j_*\mathcal{E} = \mathcal{D} \cdot (f^k \cdot e)$ .  $\square$*

We return now to the setup of Proposition 2.5.2 and put  $\mathcal{E}_k := j^*(\mathcal{D}_k/\mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi})$ , so  $\mathcal{E} = \mathcal{E}_0$ .

**Lemma 2.6.2.** *For all  $k \ll 0$ , the canonical map  $\mathcal{D}_k/\mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi} \rightarrow j_*\mathcal{E}_k$  is surjective.*

*Proof.* By adjunction, there is a canonical map

$$\mathcal{D}/\mathcal{D} \mathfrak{g} \rightarrow j_* j^*(\mathcal{D}/\mathcal{D} \mathfrak{g}) =: j_*\mathcal{E}.$$

Let  $e = e_0 \in j_*\mathcal{E}$  be the image of the class of  $1 \in \mathcal{D}$ . Clearly, we have  $\mathcal{D}_U \cdot e = \mathcal{E}$ .

For any  $k \in \mathbb{Z}$ , let  $e_k := s^k \cdot e \in j_*\mathcal{E}_k$ . Since  $\mathcal{D}_U \cdot e_0 = \mathcal{E}$ , Lemma 2.6.1 implies that there exists  $k_0 < 0$  such that, for all  $k < k_0$ , one has  $\mathcal{D}_k \cdot e_k = j_*\mathcal{E}_k$ .

The element  $e_k$  is clearly annihilated by the action of the ideal  $\mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi} \subset \mathcal{D}_k$ . Therefore, the assignment  $1 \mapsto e_k$  gives a well defined map

$$q_k : \mathcal{D}_k / \mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi} \longrightarrow j_*\mathcal{E}_k.$$

But,  $e_k$  is a generator of the  $\mathcal{D}_k$ -module  $j_*\mathcal{E}_k$ . Therefore, the map  $q_k$  is surjective, and we are done.  $\square$

For any  $k \in \mathbb{Z}$ , the TDO  $\mathcal{D}_k$  comes equipped with an increasing filtration such that  $\mathbf{gr} \mathcal{D}_k \cong p_* \mathcal{O}_{T^*X}$ , where  $p : T^*X \rightarrow X$  is the bundle projection. Let  $[SS(\mathcal{M})]$  denote the characteristic cycle of a holonomic  $\mathcal{D}_k$ -module, resp.  $[\text{supp}(\mathcal{F})]$  denote the support cycle of a coherent sheaf on  $T^*X$ .

**Lemma 2.6.3.** *Conditions (2.4.2)-(2.4.3) imply that  $\mathcal{D}_k / \mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi}$  is a holonomic  $\mathcal{D}_k$ -module and one has an equality of Lagrangian cycles*

$$[SS(\mathcal{D}_k / \mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi})] = [SS(j_*\mathcal{E}_k)], \quad \forall k \in \mathbb{Z}.$$

*Proof.* Given a pair of Lagrangian algebraic cycles  $Y, Y' \subset T^*X$ , we write  $Y \leq Y'$  whenever the cycle  $Y' - Y$  is a nonnegative integer combination of irreducible Lagrangian subvarieties.

The filtration on  $\mathcal{D}_k$  induces a natural filtration on the  $\mathcal{D}_k$ -module  $\mathcal{D}_k / \mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi}$ . One has a canonical surjection of graded  $\mathcal{O}_X$ -modules:  $\mathbf{gr} \mathcal{D}_k / (\mathbf{gr} \mathcal{D}_k) \mathfrak{g} \twoheadrightarrow \mathbf{gr}(\mathcal{D}_k / \mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi})$ . Furthermore, there is a natural isomorphism  $\mathbf{gr} \mathcal{D}_k / (\mathbf{gr} \mathcal{D}_k) \mathfrak{g} \cong p_*(\mathcal{O}_{T^*X}|_{\mu^{-1}(0)})$ . Thus, we deduce

$$[SS(\mathcal{D}_k / \mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi})] \leq [\text{supp}(\mathcal{O}_{T^*X}|_{\mu^{-1}(0)})] = [\mu^{-1}(0)].$$

Hence, Lemmas 2.4.1 and 2.4.6 imply that  $\mathcal{D}_k / \mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi}$  is a holonomic  $\mathcal{D}_k$ -module and one has

$$[SS(\mathcal{D}_k / \mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi})] \leq [\mu^{-1}(0)] = [\boldsymbol{\mu}^{-1}(0) \cap \boldsymbol{\nu}^{-1}(0)] = \lim_{a \rightarrow 0} [\boldsymbol{\mu}^{-1}(a \cdot \phi) \cap \boldsymbol{\nu}^{-1}(a)]. \quad (2.6.4)$$

Next recall that, according to [G1], Theorem 6.3, one has an equality of Lagrangian cycles  $[SS(j_*\mathcal{E}_k)] = \lim_{a \rightarrow 0} [\boldsymbol{\mu}^{-1}(a \cdot \phi) \cap \boldsymbol{\nu}^{-1}(a)]$ . Thus, from (2.6.4) we deduce

$$[SS(\mathcal{D}_k / \mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi})] \leq [SS(j_*\mathcal{E}_k)].$$

On the other hand, Lemma 2.6.2 yields an opposite inequality, and we are done.  $\square$

To complete the proof of Proposition 2.5.2, we observe that the characteristic cycle of the kernel of the surjective  $\mathcal{D}_k$ -module map  $\mathcal{D}_k / \mathcal{D}_k \mathfrak{g}^{\psi+k \cdot \phi} \twoheadrightarrow j_*\mathcal{E}_k$  (for  $k \ll 0$ ) is equal to zero, due to Lemma 2.6.3. Hence the kernel itself is zero, and we are done.  $\square$

### 3. A TRIPLE FLAG VARIETY.

**3.1. Horocycle spaces.** Let  $G$  be a connected complex semisimple group. Let  $\mathbb{T}$  be the abstract Cartan torus of  $G$ , and  $W$  the corresponding abstract Weyl group of  $G$ . Fix a Borel subgroup  $B \subset G$ , with the unipotent radical  $N$ . Thus, we have  $\mathbb{T} = B/N$ .

Let  $\mathcal{B} = G/B$  be the flag variety of  $G$ . There is a canonical  $G$ -equivariant  $\mathbb{T}$ -torsor  $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$ , where  $\tilde{\mathcal{B}} = G/N$  is the base affine space, and where  $\mathbb{T}$  acts on  $G/N$  on the right. The  $G$ - and  $\mathbb{T}$ -actions on  $\tilde{\mathcal{B}}$  commute, hence, make  $\tilde{\mathcal{B}}$  a  $G \times \mathbb{T}$ -variety.

Given elements  $x, y \in W$ , let  $\mathbb{T}_{x,y}$  be the image of the torus imbedding  $\mathbb{T} \hookrightarrow \mathbb{T} \times \mathbb{T}$ ,  $t \mapsto x(t) \times y(t)$ . The horocycle space associated with the pair  $(x, y)$  is defined to be  $\tilde{\mathcal{B}}_{x,y} := (\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) / \mathbb{T}_{x,y}$ . There are two natural projections  $\text{pr}_x, \text{pr}_y : \tilde{\mathcal{B}}_{x,y} \rightarrow \mathcal{B}$ , on the first and second factor, respectively. The left



$G \times G$ -action and the right  $\mathbb{T} \times \mathbb{T}$ -action make  $\tilde{\mathcal{B}}_{x,y}$  a smooth  $G \times G \times \mathbb{T} \times \mathbb{T}$ -variety. The right  $\mathbb{T} \times \mathbb{T}$ -action clearly factors through the torus  $(\mathbb{T} \times \mathbb{T})/\mathbb{T}_{x,y}$ . This makes the map  $\text{pr}_x \times \text{pr}_y : \tilde{\mathcal{B}}_{x,y} \twoheadrightarrow \mathcal{B} \times \mathcal{B}$  a  $G \times G$ -equivariant  $(\mathbb{T} \times \mathbb{T})/\mathbb{T}_{x,y}$ -torsor.

3.2. Write  $\mathfrak{g}, \mathfrak{t}$  for the Lie algebras of the groups  $G$  and  $\mathbb{T}$ , respectively. Let  $\mathcal{U}\mathfrak{g}$  and  $\mathcal{U}\mathfrak{t}$  be the corresponding enveloping algebras. Let  $\mathfrak{Z}$  denote the center of  $\mathcal{U}\mathfrak{g}$ . We may (and will) view  $\mathcal{U}\mathfrak{t}$  as a  $\mathfrak{Z}$ -module via the Harish-Chandra homomorphism  $\Xi : \mathfrak{Z} \rightarrow \mathcal{U}\mathfrak{t}$ .

The differential of the  $G \times \mathbb{T}$ -action on  $\tilde{\mathcal{B}}$  induces an algebra map  $\kappa : \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{t} \rightarrow \Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})$ . It is clear that the image of  $\kappa$  is contained in  $\Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})^{\mathbb{T}}$ , the subalgebra of right  $\mathbb{T}$ -invariant differential operators. It is also easy to see that, for any  $z \in \mathfrak{Z}$ , one has  $\kappa(z \otimes 1) = \kappa(1 \otimes \Xi(z))$ . Furthermore, it was shown in [BoBr] that the map  $\kappa$  gives rise to an algebra isomorphism

$$\kappa : \mathcal{U}\mathfrak{g} \otimes_{\mathfrak{Z}} \mathcal{U}\mathfrak{t} \xrightarrow{\sim} \Gamma(\tilde{\mathcal{B}}, \mathcal{D}_{\tilde{\mathcal{B}}})^{\mathbb{T}}. \quad (3.2.1)$$

Let  $\varpi_y$  be the pull-back of  $\varpi_{\mathcal{B}}$ , the canonical bundle on the flag manifold  $\mathcal{B}$ , via the projection  $\text{pr}_y : \tilde{\mathcal{B}}_{x,y} \twoheadrightarrow \mathcal{B}$ , and define

$$\mathcal{D}_{x,y} := \varpi_y \otimes_{\mathcal{O}_{\tilde{\mathcal{B}}_{x,y}}} \mathcal{D}_{\tilde{\mathcal{B}}_{x,y}} \otimes_{\mathcal{O}_{\tilde{\mathcal{B}}_{x,y}}} \varpi_y^{-1}. \quad (3.2.2)$$

It is clear that  $\mathcal{D}_{x,y}$  is a TDO on  $\tilde{\mathcal{B}}_{x,y}$ , moreover, the pair  $(\mathcal{D}_{x,y}, G \times G \times \mathbb{T} \times \mathbb{T})$  is a Harish-Chandra algebra. Thus, one has a canonical algebra map  $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t} \rightarrow \Gamma(\tilde{\mathcal{B}}_{x,y}, \mathcal{D}_{x,y})$ . We let  $\mathfrak{T}_{x,y}$  denote an ideal of the algebra  $\mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t} = \mathcal{U}(\mathfrak{t} \oplus \mathfrak{t})$  generated by  $\mathfrak{t}_{x,y} := \text{Lie } \mathbb{T}_{x,y}$ , a vector subspace of  $\mathfrak{t} \oplus \mathfrak{t} \subset \mathcal{U}(\mathfrak{t} \oplus \mathfrak{t})$ .

From the Borho-Brylinski isomorphism (3.2.1) one derives an algebra isomorphism

$$\kappa : (\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{Z} \otimes \mathfrak{Z}} [(\mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t})/\mathfrak{T}_{x,y}] \xrightarrow{\sim} \Gamma(\tilde{\mathcal{B}}_{x,y}, \mathcal{D}_{x,y})^{\mathbb{T} \times \mathbb{T}}. \quad (3.2.3)$$

A  $\mathcal{D}_{x,y}$ -module is said to be *monodromic* provided it is a holonomic  $\mathcal{D}$ -module which is, moreover,  $G$ -equivariant with respect to the  $G$ -diagonal left action on  $\tilde{\mathcal{B}}_{x,y}$  and is also monodromic with respect to the right  $\mathbb{T} \times \mathbb{T}$ -action on  $\tilde{\mathcal{B}}_{x,y}$ .

- Let  $\text{Mon}(\tilde{\mathcal{B}}_{x,y})$  denote the full abelian subcategory of  $\mathcal{D}_{x,y}\text{-mod}$  whose objects are monodromic modules.

A complex  $\mathcal{M}^* \in D^b(\mathcal{D}_{x,y}\text{-mod})$  is said to be monodromic provided, for each integer  $\ell \in \mathbb{Z}$ , the cohomology sheaf  $\mathcal{H}^\ell(\mathcal{M}^*)$  is a monodromic  $\mathcal{D}_{x,y}$ -module.

Write  $A\text{-mod}$ , resp.  $D^b(A\text{-mod})$ , for the abelian category, resp. bounded derived category, of left modules over  $A$ , an associative algebra or a sheaf of associative algebras.

- Let  $\mathbf{Dmon}(\tilde{\mathcal{B}}_{x,y})$  denote the full triangulated subcategory of  $D^b(\mathcal{D}_{x,y}\text{-mod})$  whose objects are monodromic complexes.

3.3. **In the rest of the paper we let  $G = SL_n$ .** We put  $\mathfrak{g} := \text{Lie } G = \mathfrak{sl}_n$ .

The group  $G$  acts naturally on the projective space  $\mathbb{P} = \mathbb{P}(\mathbb{C}^n)$ , and we let  $G$  act diagonally on  $\mathcal{B} \times \mathcal{B} \times \mathbb{P}$ , a ‘triple flag manifold’. There is a moment map associated with the induced Hamiltonian  $G$ -action on  $T^*(\mathcal{B} \times \mathcal{B} \times \mathbb{P})$ , the total space of the cotangent bundle.

The crucial geometric properties of the triple flag manifold are summarized in the following proposition.

**Proposition 3.3.1.** (i) *The group  $G$  acts diagonally on  $\mathcal{B} \times \mathcal{B} \times \mathbb{P}$  with finitely many orbits. The orbits are parametrized by the set of pairs  $(w, \sigma)$ , where  $w \in \mathbb{S}_n$  is a permutation and  $\sigma$  is a decreasing subsequence of the sequence of integers  $w(1), \dots, w(n)$ .*

(ii) *The moment map  $\mu : T^*(\mathcal{B} \times \mathcal{B} \times \mathbb{P}) \rightarrow \mathfrak{g}^*$  is flat.*

*Proof.* Part (i) is proved in [MWZ], 2.11. Part (ii) follows from Lemma 2.4.6(i)-(ii) and the equation

$$\dim(\mathcal{B} \times \mathcal{B} \times \mathbb{P}) = \frac{n(n-1)}{2} + \frac{n(n-1)}{2} + (n-1) = n^2 - 1 = \dim G. \quad \square$$

Next, we fix a pair of elements  $x, y \in W$  and put  $\mathfrak{B}_{x,y} = \tilde{\mathcal{B}}_{x,y} \times \mathbb{P}$ . Below, we will be interested in the left  $G$ -diagonal action on  $\mathfrak{B}_{x,y}$  as well as in the right  $\mathbb{T} \times \mathbb{T}$ -action induced by the one on  $\tilde{\mathcal{B}}_{x,y}$ , the first factor, and trivial along  $\mathbb{P}$ , the second factor. The right  $\mathbb{T} \times \mathbb{T}$ -action clearly factors through a  $(\mathbb{T} \times \mathbb{T})/\mathbb{T}_{x,y}$ -action. It follows that the left  $G$ -action factors through the adjoint group  $G/\text{Center}(G)$ . These left and right actions make  $\mathfrak{B}_{x,y}$  a  $G \times \mathbb{T} \times \mathbb{T}$ -variety, and there is an induced Hamiltonian  $G \times \mathbb{T} \times \mathbb{T}$ -action on the cotangent bundle  $T^*\mathfrak{B}_{x,y}$ .

Let  $\mathfrak{t}_{x,y}^\perp \subset \mathfrak{t}^* \oplus \mathfrak{t}^*$  denote the annihilator of the subspace  $\mathfrak{t}_{x,y} \subset \mathfrak{t} \oplus \mathfrak{t}$ . It is immediate to see that the image of an associated moment map is automatically contained in the subspace  $\mathfrak{g}^* \times \mathfrak{t}_{x,y}^\perp \subset \mathfrak{g}^* \times \mathfrak{t}^* \times \mathfrak{t}^*$ .

From Proposition 3.3.1 one easily deduces the following

**Corollary 3.3.2.** (i) *The group  $G \times \mathbb{T} \times \mathbb{T}$  acts on  $\mathfrak{B}_{x,y}$  with finitely many orbits.*

(ii) *The moment map  $\mu : T^*\mathfrak{B}_{x,y} \longrightarrow \mathfrak{g}^* \times \mathfrak{t}_{x,y}^\perp$  is flat.*  $\square$

3.4. For any  $c \in \mathbb{C}$ , let  $\mathcal{D}_{\mathbb{P}}^c$  denote the sheaf of TDO on  $\mathbb{P}$  with twist ‘ $c$ ’. Given  $x, y \in W$ , we write  $\mathcal{D}_{x,y}^c := \mathcal{D}_{x,y} \boxtimes \mathcal{D}_{\mathbb{P}}^c$  for the corresponding TDO on  $\mathfrak{B}_{x,y} = \tilde{\mathcal{B}}_{x,y} \times \mathbb{P}$ . Thus,  $(\mathcal{D}_{x,y}^c, G \times \mathbb{T} \times \mathbb{T})$  is a Harish-Chandra algebra.

We have the notion of a monodromic  $\mathcal{D}_{x,y}^c$ -module, resp. monodromic complex, defined the same way as we have done in §3.2, with the space  $\tilde{\mathcal{B}}_{x,y}$  being now replaced by  $\mathfrak{B}_{x,y}$ . Similarly to section 3.2, one defines an abelian category  $\text{Mon}_c(\mathfrak{B}_{x,y}) \subset \mathcal{D}_{x,y}^c\text{-mod}$ , of monodromic  $\mathcal{D}_{x,y}^c$ -modules, resp. triangulated category  $\text{Dmon}_c(\mathfrak{B}_{x,y}) \subset D^b(\mathcal{D}_{x,y}^c\text{-mod})$ , of monodromic complexes.

Corollary 3.3.2(i) implies the following

**Corollary 3.4.1.** *Any object of  $\text{Mon}_c(\mathfrak{B}_{x,y})$  is a holonomic  $\mathcal{D}$ -module with regular singularities.*

Let  $\mathfrak{t}_{\mathbb{Z}}^* \subset \mathfrak{t}^*$  denote the weight lattice of  $\mathbb{T}$  and, given  $x, y \in W$ , write  $\text{pr}_{x,y} = \text{pr}_x \times \text{pr}_y \times \text{Id}_{\mathbb{P}} : \mathfrak{B}_{x,y} = \tilde{\mathcal{B}}_{x,y} \times \mathbb{P} \rightarrow \mathcal{B} \times \mathcal{B} \times \mathbb{P}$  for the natural projection.

For any monodromic complex  $\mathcal{V} \in \text{Dmon}_c(\mathfrak{B}_{x,y})$  there is a natural monodromy action of the fundamental group of the torus  $\mathbb{T} \times \mathbb{T}$  in the stalks of  $(\text{pr}_{x,y})_* \mathcal{H}^\bullet(\mathcal{V})$ , a sheaf theoretic direct image of the cohomology of  $\mathcal{V}$ . Note that one may view the fundamental group of  $\mathbb{T} \times \mathbb{T}$  as a lattice  $\pi_1(\mathbb{T} \times \mathbb{T}) \subset \mathfrak{t} \oplus \mathfrak{t}$ .

Let  $\bar{\lambda}, \bar{\nu} \in \mathfrak{t}^*/\mathfrak{t}_{\mathbb{Z}}^*$ . We say that  $\mathcal{V} \in \text{Dmon}_c(\mathfrak{B}_{x,y})$  has monodromy  $(\bar{\lambda}, \bar{\nu})$  provided, for any element  $(t_1, t_2) \in \pi_1(\mathbb{T} \times \mathbb{T}) \subset \mathfrak{t} \times \mathfrak{t}$ , the corresponding monodromy  $(t_1, t_2)$ -action in  $(\text{pr}_{x,y})_* \mathcal{H}^\bullet(\mathcal{V})$  is a linear operator of the form  $e^{2\pi\sqrt{-1}(\langle \lambda, t_1 \rangle + \langle \nu, t_2 \rangle)} \text{Id} + u$ , where  $u$  is a nilpotent operator. In this formula,  $\langle \lambda, t_1 \rangle$ , resp  $\langle \nu, t_2 \rangle$ , denotes the value at  $t_i \in \mathfrak{t}$  of an arbitrary representative in  $\mathfrak{t}^*$  of the corresponding element  $\bar{\lambda}, \bar{\nu} \in \mathfrak{t}^*/\mathfrak{t}_{\mathbb{Z}}^*$ .

- Let  $\text{Mon}_{\bar{\lambda}, \bar{\nu}, c}(\mathfrak{B}_{x,y})$  be the full abelian subcategory of  $\text{Mon}_c(\mathfrak{B}_{x,y})$ , resp.  $\text{Dmon}_{\bar{\lambda}, \bar{\nu}, c}(\mathfrak{B}_{x,y})$  be the full triangulated subcategory of  $\text{Dmon}_c(\mathfrak{B}_{x,y})$ , whose objects have monodromy  $(\bar{\lambda}, \bar{\nu})$ .

It is clear that the category  $\text{Dmon}_{\bar{\lambda}, \bar{\nu}, c}(\mathfrak{B}_{x,y})$  is trivial unless the character  $(\bar{\lambda}, \bar{\nu})$  restricts to the unit character of the subgroup  $\pi_1(\mathbb{T}_{x,y}) \subset \pi_1(\mathbb{T} \times \mathbb{T})$ . Let  $\bar{\mathfrak{t}}_{x,y}^\perp \subset \mathfrak{t}^*/\mathfrak{t}_{\mathbb{Z}}^* \oplus \mathfrak{t}^*/\mathfrak{t}_{\mathbb{Z}}^*$  be the set of such characters, i.e., the set pairs  $(\bar{\lambda}, \bar{\nu})$  such that the linear function  $(\lambda, \nu)$  takes integer values on the lattice  $\pi_1(\mathbb{T}_{x,y})$ , for some (equivalently, any) representative  $(\lambda, \nu) \in \mathfrak{t}^* \oplus \mathfrak{t}^*$  of the element  $(\bar{\lambda}, \bar{\nu})$ .

Thus, there is a canonical direct sum decomposition

$$\text{Mon}_c(\mathfrak{B}_{x,y}) = \bigoplus_{(\bar{\lambda}, \bar{\nu}) \in \bar{\mathfrak{t}}_{x,y}^\perp} \text{Mon}_{\bar{\lambda}, \bar{\nu}, c}(\mathfrak{B}_{x,y}), \quad \text{resp.} \quad \text{Dmon}_c(\mathfrak{B}_{x,y}) = \bigoplus_{(\bar{\lambda}, \bar{\nu}) \in \bar{\mathfrak{t}}_{x,y}^\perp} \text{Dmon}_{\bar{\lambda}, \bar{\nu}, c}(\mathfrak{B}_{x,y}).$$

For any object  $\mathcal{V} \in \mathbf{Dmon}_c(\mathfrak{B}_{x,y})$ , we write  $\mathcal{V} = \oplus_{\bar{\lambda}, \bar{\nu}} \mathcal{V}^{(\bar{\lambda}, \bar{\nu})}$  for the corresponding direct sum decomposition.

**3.5. Convolution.** Let  $\rho \in \mathfrak{t}^*$  denote the half-sum of positive roots. Associated with any triple  $x, y, z \in W$ , there is a standard convolution functor, cf. [BG], §5:

$$* : \mathbf{Dmon}_{\bar{\lambda}, \bar{\nu}, c}(\mathfrak{B}_{x,y}) \times \mathbf{Dmon}_{-\bar{\nu}, \bar{\lambda}', c'}(\mathfrak{B}_{y,z}) \longrightarrow \mathbf{Dmon}_{\bar{\lambda}, \bar{\lambda}', c+c'}(\mathfrak{B}_{x,z}). \quad (3.5.1)$$

To define convolution (3.5.1), one writes  $\mathbb{T}_{x,y,z}$  for the image of a torus imbedding  $\mathbb{T} \hookrightarrow \mathbb{T} \times \mathbb{T} \times \mathbb{T}$ , given by  $t \mapsto x(t) \times y(t) \times z(t)$ . Clearly,  $(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})/\mathbb{T}_{x,y,z}$  is a smooth variety and there are 3 natural projections along various factors  $\tilde{\mathcal{B}}$ , eg.  $\mathbf{q}_{x,y} : (\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})/\mathbb{T}_{x,y,z} \rightarrow \tilde{\mathcal{B}}_{x,y}$ . We may extend each of these morphisms to a cartesian product with a copy of the projective space  $\mathbb{P}$ . This way, we get e.g. a map  $\mathbf{q}_{x,y} \boxtimes \text{Id}_{\mathbb{P}} : [(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})/\mathbb{T}_{x,y,z}] \times \mathbb{P} \rightarrow \tilde{\mathcal{B}}_{x,y} \times \mathbb{P}$ , and also a map  $\mathbf{q}_y \boxtimes \text{Id}_{\mathbb{P}}$ .

There is also a natural map  $\mathbf{q}_y : (\tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \tilde{\mathcal{B}})/\mathbb{T}_{x,y,z} \rightarrow \tilde{\mathcal{B}}$ , the projection to the middle factor. The definition of the TDO  $\mathcal{D}_{x,y}$ , see (3.2.2), combined with the canonical isomorphism  $\mathcal{D}_{\mathcal{B}}^{\text{op}} \cong \varpi_{\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{D}_{\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{B}}} \varpi_{\mathcal{B}}^{-1}$  insure that, for any  $\mathcal{D}_{x,y}$ -module  $M$ , the sheaf  $\mathbf{q}_{x,y}^* M$  has a natural right  $\mathbf{q}_y^* \mathcal{D}_{\tilde{\mathcal{B}}}$ -module structure.

In formula (3.5.2) below, we use simplified notation and write  $\mathbf{q}_{x,y}$  for  $\mathbf{q}_{x,y} \boxtimes \text{Id}_{\mathbb{P}}$ . With that understood, the convolution of any pair of objects  $\mathcal{M} \in \mathbf{Dmon}_{\bar{\lambda}, \bar{\nu}, c}(\mathfrak{B}_{x,y})$  and  $\mathcal{N} \in \mathbf{Dmon}_{-\bar{\nu}, \bar{\lambda}', c'}(\mathfrak{B}_{y,z})$  is defined by

$$\mathcal{M} * \mathcal{N} := (R\mathbf{q}_{x,z})_{\bullet} (\mathbf{q}_{x,y}^* \mathcal{M} \overset{L}{\bigotimes}_{(\mathbf{q}_y^* \mathcal{D}_{\tilde{\mathcal{B}}}) \boxtimes \mathcal{O}_{\mathbb{P}}} \mathbf{q}_{y,z}^* \mathcal{N}). \quad (3.5.2)$$

**3.6.** Let  $w_0 \in W$  denote the element of maximal length. The variety  $\tilde{\mathcal{B}}_{1,w_0}$  will play a special role. This variety has a  $G$ -equivariantly trivialized canonical bundle. Note also that, for any  $y \in W$ , we have  $\tilde{\mathcal{B}}_{y,w_0y} = \tilde{\mathcal{B}}_{1,w_0}$ , hence  $\mathfrak{B}_{y,w_0y} = \mathfrak{B}_{1,w_0}$ .

We are going to introduce, following [BG], §5, a pair of pro-objects of the category  $\mathcal{D}_{y,w_0y}$ -mod that will be important in subsequent sections. To this end, view  $\tilde{\mathcal{B}}_{y,w_0y}$  as a  $G \times \mathbb{T} \times \mathbb{T}$ -variety where  $G$ , the first factor, acts diagonally on  $\tilde{\mathcal{B}}_{y,w_0y}$  on the left, and the group  $\mathbb{T} \times \mathbb{T}$  acts naturally on the right. Let  $\mathcal{U}$  be the unique Zariski open dense  $G \times \mathbb{T} \times \mathbb{T}$ -orbit in  $\tilde{\mathcal{B}}_{y,w_0y}$ . We have the following diagram involving natural projections and an open imbedding:

$$G \backslash \mathcal{U} \xleftarrow{h} \mathcal{U} \xhookrightarrow{j} \tilde{\mathcal{B}}_{y,w_0y} \xleftarrow{\text{pr}_1} \mathfrak{B}_{y,w_0y} = \tilde{\mathcal{B}}_{y,w_0y} \times \mathbb{P}. \quad (3.6.1)$$

For any  $u \in \mathcal{U}$ , the isotropy group of  $u$  with respect to the left  $G$ -diagonal action is a maximal torus in  $G$ , in particular, it is connected. We have an isomorphism  $(\mathbb{T} \times \mathbb{T})/\mathbb{T}_{y,w_0y} \xrightarrow{\sim} \mathbb{T}$ ,  $(t_1, t_2) \mapsto t_1$ . The right  $\mathbb{T} \times \mathbb{T}$ -action on  $\mathcal{U}$  descends to a well defined  $(\mathbb{T} \times \mathbb{T})/\mathbb{T}_{y,w_0y}$ -action on the orbit space  $G \backslash \mathcal{U}$  and makes the latter space a  $\mathbb{T}$ -torsor.

Define a linear involution

$$(-)^{\dagger} : \mathfrak{t}^* \rightarrow \mathfrak{t}^*, \quad \lambda \mapsto \lambda^{\dagger} := -w_0 \lambda. \quad (3.6.2)$$

Given  $\lambda \in \mathfrak{t}^*$ , let  $\mathcal{J}^{\lambda}$  be a rank 1 local system on the  $\mathbb{T}$ -torsor  $G \backslash \mathcal{U}$  with monodromy  $e^{2\pi\sqrt{-1}\lambda}$ . Thus,  $h^*(\mathcal{J}^{\lambda})$ , cf. (3.6.1), is a local system on  $\mathcal{U}$  with monodromy  $(\lambda, \lambda^{\dagger}) \in \mathfrak{t}^* \times \mathfrak{t}^*$ . Further, let  $\mathcal{E}^{\infty}$  be a pro-unipotent local system on  $G \backslash \mathcal{U}$ , cf. [BG], page 18. Such a local system is unique up to isomorphism, and we put  $\hat{\mathcal{J}}^{\lambda} := \mathcal{J}^{\lambda} \otimes \mathcal{E}^{\infty}$ . All stabilizers of the  $G$ -action in  $\mathcal{U}$  being connected, it follows that  $h^*(\hat{\mathcal{J}}^{\lambda})$  is a (projective limit of)  $G$ -equivariant local systems on  $\mathcal{U}$ .

We use diagram (3.6.1) to define the following pair of (projective limits of)  $G$ -equivariant holonomic  $\mathcal{D}_{y,w_0y}$ -modules (cf. also [BG, formulas (5.9.2), (5.22)]):

$$\mathcal{R}_*^{\lambda, \lambda^{\dagger}} := \text{pr}_1^* \circ j_* \circ h^*(\hat{\mathcal{J}}^{\lambda}), \quad \mathcal{R}_!^{\lambda, \lambda^{\dagger}} := \text{pr}_1^* \circ j_! \circ h^*(\hat{\mathcal{J}}^{\lambda}) \in \lim \text{proj Mon}_{\bar{\lambda}, \bar{\lambda}^{\dagger}, 0}(\mathfrak{B}_{y,w_0y}). \quad (3.6.3)$$

For any  $x, y \in W$  and  $\nu \in \mathfrak{t}^*$ , convolution with the above objects yields the following *mutually inverse* (triangulated) equivalences, see [BG] §5:

$$\begin{aligned} \mathbf{Dmon}_{\bar{\nu}, -\bar{\lambda}, c}(\mathfrak{B}_{x, y}) &\longrightarrow \mathbf{Dmon}_{\bar{\nu}, \bar{\lambda}^\dagger, c}(\mathfrak{B}_{x, w_0 y}), & \mathcal{M} &\mapsto \mathcal{M} * \mathcal{R}_!^{\lambda, \lambda^\dagger} \\ \mathbf{Dmon}_{\bar{\nu}, \bar{\lambda}^\dagger, c}(\mathfrak{B}_{x, w_0 y}) &\longrightarrow \mathbf{Dmon}_{\bar{\nu}, -\bar{\lambda}, c}(\mathfrak{B}_{x, y}), & \mathcal{V} &\mapsto \mathcal{V} * \mathcal{R}_*^{-\lambda^\dagger, -\lambda}. \end{aligned} \quad (3.6.4)$$

#### 4. MIRABOLIC $\mathscr{D}$ -MODULES

**4.1. Character  $\mathscr{D}$ -modules.** Recall that  $G = SL_n$  and write  $\mathscr{D}_G$  for the sheaf of differential operators on  $G$ .

The group  $G \times G$  acts on  $G$  by left and right translations, via  $(g_1 \times g_2) : g \mapsto g_1 g g_2^{-1}$ . The differential of that action induces a linear map from  $\mathfrak{g} \times \mathfrak{g}$  to the Lie algebra of vector fields on  $G$ . The latter map extends uniquely to an associative algebra homomorphism  $\mathrm{lr} : \mathcal{U}\mathfrak{g} \otimes (\mathcal{U}\mathfrak{g})^{\mathrm{op}} \rightarrow \mathscr{D}(G)$ .

It is clear that the sets  $\mathrm{lr}(\mathfrak{z} \otimes 1)$ ,  $\mathrm{lr}(1 \otimes \mathfrak{z})$  are contained in  $\mathscr{D}(G)^{G \times G} \subset \mathscr{D}(G)$ , the subalgebra of *bi-invariant* differential operators on  $G$ . In fact, it is easy to see that one has  $\mathrm{lr}(z \otimes 1) = \mathrm{lr}(1 \otimes \tau(z))$ , where  $\tau : \mathcal{U}\mathfrak{g} \xrightarrow{\sim} (\mathcal{U}\mathfrak{g})^{\mathrm{op}}$  is an algebra isomorphism defined on generators  $x \in \mathfrak{g}$  by  $\tau(x) = -x$ . In this way, one obtains a well defined, injective algebra morphism

$$\mathrm{lr} \circ (\mathrm{Id} \times \tau) : \mathcal{U}\mathfrak{g} \otimes_{\mathfrak{z}} \mathcal{U}\mathfrak{g} \hookrightarrow \mathscr{D}(G), \quad u \otimes u' \mapsto \mathrm{lr}(u \otimes \tau(u')).$$

Next, we put  $\mathfrak{X} := G \times \mathbb{P}$ . Recall that  $\mathscr{D}_{\mathbb{P}}^c$  stands for the sheaf of TDO on  $\mathbb{P} = \mathbb{P}^{n-1}$ , with twist  $c \in \mathbb{C}$ . We let  $\mathscr{D}_{\mathfrak{X}}^c := \mathscr{D}_G \boxtimes \mathscr{D}_{\mathbb{P}}^c$  be an associated TDO on  $\mathfrak{X}$ , and write  $\mathscr{D}(\mathbb{P}, c) := \Gamma(\mathbb{P}, \mathscr{D}_{\mathbb{P}}^c)$ , resp.  $\mathscr{D}(\mathfrak{X}, c) = \Gamma(\mathfrak{X}, \mathscr{D}_{\mathfrak{X}}^c) = \Gamma(G, \mathscr{D}_G) \otimes \mathscr{D}(\mathbb{P}, c)$ , for the corresponding algebras of global sections. Let  $\zeta$  denote the composite of the following algebra maps

$$\zeta : (\mathcal{U}\mathfrak{g} \otimes_{\mathfrak{z}} \mathcal{U}\mathfrak{g}) \otimes \mathscr{D}(\mathbb{P}, c) \xrightarrow{[\mathrm{lr} \circ (\mathrm{Id} \otimes \tau)] \otimes \mathrm{Id}} \mathscr{D}(G) \otimes \mathscr{D}(\mathbb{P}, c) = \mathscr{D}(\mathfrak{X}, c). \quad (4.1.1)$$

Further, we have an obvious imbedding  $v : \mathfrak{z} \hookrightarrow (\mathcal{U}\mathfrak{g} \otimes_{\mathfrak{z}} \mathcal{U}\mathfrak{g}) \otimes \mathscr{D}(\mathbb{P}, c)$ ,  $z \mapsto (z \otimes 1) \otimes 1 = (1 \otimes z) \otimes 1$ .

Recall that an action of an algebra  $A$  on an  $A$ -module  $M$  is said to be *locally finite* if, for any  $m \in M$ , one has  $\dim A \cdot m < \infty$ .

Now, we let  $G$  act on itself via the adjoint action, and let  $G$  act diagonally on  $\mathfrak{X} = G \times \mathbb{P}$ .

**Definition 4.1.2.** A  $\mathscr{D}_{\mathfrak{X}}^c$ -module  $\mathcal{F}$  is called a (mirabolic) *character module* if the following two conditions hold, cf. [GG], Remark 5.2, and also [FG], §§4-5:

- (1)  $\mathcal{F}$  is a  $G$ -equivariant  $\mathscr{D}$ -module with respect to the  $G$ -diagonal action on  $\mathfrak{X} = G \times \mathbb{P}$ .
  - (2) The left  $\mathfrak{z}$ -action on  $\Gamma(\mathfrak{X}, \mathcal{F})$ , via the imbedding  $\zeta \circ v$ , is locally finite.
- Let  $\mathscr{C}_c(\mathfrak{X})$  be the full abelian subcategory of  $\mathscr{D}_{\mathfrak{X}}^c\text{-mod}$  whose objects are character modules.

**4.2. Spectral decomposition.** Given a commutative algebra  $A$ , write  $\mathrm{Max}(A)$  for the set of maximal ideals in  $A$ . For  $\mathfrak{a} \in \mathrm{Max}(A)$ , and an  $A$ -module  $M$ , we put

$$M^{(\mathfrak{a})} := \{m \in M \mid \exists \ell = \ell(m) \gg 0 \text{ such that } \mathfrak{a}^\ell \cdot m = 0\}. \quad (4.2.1)$$

If, moreover, the  $A$ -action on  $M$  is locally finite, then one has a canonical  $A$ -stable *spectral decomposition*

$$M = \bigoplus_{\mathfrak{a} \in \mathrm{Max}(A)} M^{(\mathfrak{a})}. \quad (4.2.2)$$

First, we apply (4.2.2) in the case where  $A := \mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t}$ . Given  $\lambda \in \mathfrak{t}^*$ , let  $I_\lambda \subset \mathcal{U}\mathfrak{t}$  denote a maximal ideal generated by the elements  $\{x - \lambda(x), x \in \mathfrak{t}\}$ . Similarly, associated with any  $\lambda \times \nu \in \mathfrak{t}^* \times \mathfrak{t}^*$ , there is a maximal ideal  $I_{\lambda, \nu} := I_\lambda \otimes \mathcal{U}\mathfrak{t} + \mathcal{U}\mathfrak{t} \otimes I_\nu \in \mathrm{Max}(\mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t})$ . Thus, for a locally finite  $\mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t}$ -module  $M$ , one has a spectral decomposition  $M = \bigoplus_{\lambda \times \nu \in \mathfrak{t}^* \times \mathfrak{t}^*} M^{I_{\lambda, \nu}}$ .

Next, we take  $A = \mathfrak{Z}$ , the center of  $\mathcal{U}\mathfrak{g}$ . Given  $\theta \in \mathfrak{t}^*/W$ , let  $\mathfrak{Z}_\theta \subset \mathfrak{Z}$  be the maximal ideal corresponding to  $\theta$  via the Harish-Chandra isomorphism  $\Xi : \mathfrak{Z} \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}^*/W]$ . Thus, for any locally finite  $\mathfrak{Z}$ -module  $M$ , one has a spectral decomposition  $M = \bigoplus_{\theta \in \mathfrak{t}^*/W} M^{(\mathfrak{Z}_\theta)}$ .

Similarly, let  $M \in D^b(\mathfrak{Z}\text{-mod})$  be a complex such the induced  $\mathfrak{Z}$ -action on  $H^\bullet(M)$  is locally finite. Then, one can show that there is a canonical direct sum decomposition  $M = \bigoplus_{\theta \in \mathfrak{t}^*/W} M^{(\theta)}$  where  $M^{(\theta)} \in D^b(\mathfrak{Z}\text{-mod})$  are such that, for each  $\theta$ , one has  $H^\bullet(M^{(\theta)}) = [H^\bullet(M^{(\theta)})]^{(\mathfrak{Z}_\theta)}$ .

We write  $\mathfrak{Z}_\theta^c \subset \mathcal{D}(\mathfrak{X}, c)$  for  $\zeta \circ v(\mathfrak{Z}_\theta)$ , the image of  $\mathfrak{Z}_\theta$  under the composite imbedding  $\zeta \circ v : \mathfrak{Z} \rightarrow \mathcal{D}(\mathfrak{X}, c)$ , see (4.1.1). For any character  $\mathcal{D}$ -module  $\mathcal{F} \in \mathcal{C}_c(\mathfrak{X})$ , one has a vector space decomposition  $H^\bullet(\mathfrak{X}, \mathcal{F}) = \bigoplus_{\theta \in \mathfrak{t}^*/W} H^\bullet(\mathfrak{X}, \mathcal{F})^{(\mathfrak{Z}_\theta^c)}$ .

It will be convenient to introduce a semi-direct product  $W_{\text{aff}} := W \ltimes \mathfrak{t}_{\mathbb{Z}}^*$ , an (extended) affine Weyl group. The group  $W_{\text{aff}}$  acts naturally on  $\mathfrak{t}^*$  by affine linear transformations. Let  $\Theta \in \mathfrak{t}^*/W_{\text{aff}}$ . It is easy to see that, for any  $\mathcal{F} \in \mathcal{C}_c(\mathfrak{X})$ , the vector space

$$H^\bullet(\mathfrak{X}, \mathcal{F})\langle \Theta \rangle := \bigoplus_{\{\theta \in \mathfrak{t}^*/W \mid \theta \pmod{W_{\text{aff}}} = \Theta\}} H^\bullet(\mathfrak{X}, \mathcal{F})^{(\mathfrak{Z}_\theta^c)}$$

is a  $\mathcal{D}(\mathfrak{X}, c)$ -submodule in  $H^\bullet(\mathfrak{X}, \mathcal{F})$ . Furthermore, there is a unique  $\mathcal{D}$ -submodule  $\mathcal{F}\langle \Theta \rangle \subset \mathcal{F}$  such that, one has  $H^\bullet(\mathfrak{X}, \mathcal{F}\langle \Theta \rangle) = H^\bullet(\mathfrak{X}, \mathcal{F})\langle \Theta \rangle$ .

Let  $\mathcal{C}_{\Theta, c}(\mathfrak{X})$  be a full subcategory of  $\mathcal{C}_c(\mathfrak{X})$  whose objects satisfy  $\mathcal{F} = \mathcal{F}\langle \Theta \rangle$ . This way, following the strategy of [G2], one obtains a canonical spectral decomposition, cf. [G2], Theorem 1.3.2:

$$\mathcal{C}_c(\mathfrak{X}) = \bigoplus_{\Theta \in \mathfrak{t}^*/W_{\text{aff}}} \mathcal{C}_{\Theta, c}(\mathfrak{X}), \quad \mathcal{F} = \bigoplus_{\Theta \in \mathfrak{t}^*/W_{\text{aff}}} \mathcal{F}\langle \Theta \rangle, \quad \mathcal{F}\langle \Theta \rangle \in \mathcal{C}_{\Theta, c}(\mathfrak{X}), \quad \forall \Theta. \quad (4.2.3)$$

- Let  $\mathbf{D}\mathcal{C}_c(\mathfrak{X})$ , resp.  $\mathbf{D}\mathcal{C}_{\Theta, c}(\mathfrak{X})$ , denote the full triangulated subcategory of  $D^b(\mathcal{D}_{\mathfrak{X}}^c\text{-mod})$  whose objects are complexes  $\mathcal{F}^\bullet$  such that for the corresponding cohomology sheaves, one has  $\mathcal{H}^j(\mathcal{F}^\bullet) \in \mathcal{C}_c(\mathfrak{X})$ , resp.  $\mathcal{H}^j(\mathcal{F}^\bullet) \in \mathcal{C}_{\Theta, c}(\mathfrak{X})$  and  $\mathcal{H}^j(\mathcal{F}^\bullet) = [\mathcal{H}^j(\mathcal{F}^\bullet)]\langle \Theta \rangle$ , for any  $j \in \mathbb{Z}$ , resp. any  $\Theta \in \mathfrak{t}^*/W_{\text{aff}}$ .

**4.3. The functors CH and HC.** An important role in what follows will be played by a diagram

$$\begin{array}{ccc} & G \times (\tilde{\mathcal{B}}/\mathbb{T}) \times \mathbb{P} & \\ p \swarrow & & \searrow q \\ \mathfrak{X} = G \times \mathbb{P} & & \mathfrak{B}_{1,1} \end{array} \quad (4.3.1)$$

In this diagram, the maps  $p$  and  $q$  are given by  $p(g, \tilde{x}, l) := (g, l)$  and  $q(g, \tilde{x}, l) := (g\tilde{x}, \tilde{x}, gl)$ , respectively. The group  $G$  acts diagonally on all cartesian products involved in the above diagram, and acts on  $G$  by conjugation. With these  $G$ -actions, all maps in (4.3.1) become  $G$ -equivariant morphisms.

Recall the notation  $\varpi_y$  for the pull-back of  $\omega_{\mathcal{B}}$ , the canonical bundle on the flag manifold, via the second projection  $\tilde{\mathcal{B}}_{x,y} = (\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})/\mathbb{T}_{x,y} \rightarrow \mathcal{B}$ . Given a quasi-coherent sheaf  $\mathcal{V}$  on  $\mathfrak{B}_{x,y} = \tilde{\mathcal{B}}_{x,y} \times \mathbb{P}$ , we will often abuse the notation and write  $\varpi^{\pm 1} \otimes \mathcal{V}$  for  $(\varpi_y^{\pm 1} \boxtimes \mathcal{O}_{\mathbb{P}}) \otimes_{\mathcal{O}_{\tilde{\mathcal{B}}_{x,y} \times \mathbb{P}}} \mathcal{V}$ .

Now, fix  $\lambda \in \mathfrak{t}^*$ , and let  $\bar{\lambda} \in \mathfrak{t}^*/\mathbb{Z}$ , resp.  $\Theta \in \mathfrak{t}^*/W_{\text{aff}}$ , be the image of  $\lambda$ . We use diagram (4.3.1) and introduce a pair of functors, CH and HC, given, for any  $\mathcal{F} \in \mathbf{D}\mathcal{C}_{\Theta, c}(\mathfrak{X})$ , resp.  $\mathcal{V} \in \mathbf{D}\text{mon}_{\bar{\lambda}, \bar{\lambda}^\dagger, c}(\mathfrak{B}_{1, w_0})$ , by the formulas, cf. (3.6.4) and [G2, §8],

$$\text{HC}(\mathcal{F}) := (\varpi^{-1} \otimes q_! p^*(\mathcal{F})) * \mathcal{R}_1^{\lambda, \lambda^\dagger}[\dim \mathcal{B}], \quad \text{CH}(\mathcal{V}) := p_* q^!(\varpi \otimes (\mathcal{V} * \mathcal{R}_*^{-\lambda^\dagger, -\lambda}))[-\dim \mathcal{B}].$$

**Proposition 4.3.2.** (i) *The above formulas give an adjoint pair,  $(\text{CH}, \text{HC})$ , of triangulated functors*

$$\mathbf{D}\mathcal{C}_{\Theta,c}(\mathfrak{X}) \begin{array}{c} \xrightarrow{\text{HC}} \\ \xleftarrow{\text{CH}} \end{array} \mathbf{D}\text{mon}_{\bar{\lambda}, \bar{\lambda}^\dagger, c}(\mathfrak{B}_{1, w_0}).$$

(ii) *Any character module  $\mathcal{F} \in \mathcal{C}_{\Theta,c}$  is regular holonomic and the functor  $\text{CH} \circ \text{HC}$  contains  $\text{Id}_{\mathbf{D}\mathcal{C}_c(\mathfrak{X})}$ , the identity functor, as a direct summand.*

The proof of the proposition repeats word by word the proof of Theorem 4.4 of [MV] or [G2], §9, using Corollary 3.4.1.

We can now formulate one of the main results of the paper.

**Theorem 4.3.3.** *For any regular dominant weight  $\lambda \in \mathfrak{t}^*$  and  $c \neq -1, -2, \dots, 1 - n$ , the functor  $\text{HC}$  restricts to an exact functor  $\text{HC} : \mathcal{C}_{\Theta,c}(\mathfrak{X}) \rightarrow \text{Mon}_{\bar{\lambda}, \bar{\lambda}^\dagger, c}(\mathfrak{B}_{1, w_0})$ , between abelian categories.*

This theorem is a ‘mirabolic analogue’ of a result of Bezrukavnikov, Finkelberg, and Ostrik, [BFO].

Theorem 4.3.3 will be deduced from a more precise result, Theorem 4.4.4 of the next subsection.

4.4. We will use simplified notation  $\mathcal{D}(\mathbb{P}, c) := \Gamma(\mathbb{P}, \mathcal{D}_{\mathbb{P}}^c)$ , resp.  $\mathcal{D}(\mathfrak{X}, c) := \Gamma(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}}^c)$ , and  $\mathcal{D}(\mathfrak{B}_{x,y}, c) := \Gamma(\mathfrak{B}_{x,y}, \mathcal{D}_{x,y}^c)$ , for any  $x, y \in W$ .

An important role below will be played by the algebra

$$\mathbf{U}_c := (\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{z} \otimes \mathfrak{z}} (\mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t}) \otimes \mathcal{D}(\mathbb{P}, c). \quad (4.4.1)$$

The algebra (4.4.1) fits into a diagram of algebra maps

$$\begin{array}{ccccc} & & \mathbf{U}_c & \xleftarrow{\epsilon} & \mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t} \\ & \swarrow \eta = pr \otimes a \otimes \text{Id} & & \searrow \kappa_{x,y} \otimes \text{Id} & \\ \mathcal{D}(\mathfrak{X}, c) & \xleftarrow{\zeta} & (\mathcal{U}\mathfrak{g} \otimes_{\mathfrak{z}} \mathcal{U}\mathfrak{g}) \otimes \mathcal{D}(\mathbb{P}, c) & & \mathcal{D}(\mathfrak{B}_{x,y}, c)^{\mathbb{T} \times \mathbb{T}} \xrightarrow{j} \mathcal{D}(\mathfrak{B}_{x,y}, c). \end{array} \quad (4.4.2)$$

In this diagram, the map  $j$  is the natural inclusion, the map  $\zeta$  is the isomorphism (4.1.1), and the map  $\kappa_{x,y}$  comes from the isomorphism  $\kappa$  in (4.1.1). Further, the map  $\eta$  is induced by the augmentation  $a : \mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t} \rightarrow \mathbb{C}$  and by the natural projection  $pr : \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g} \twoheadrightarrow \mathcal{U}\mathfrak{g} \otimes_{\mathfrak{z}} \mathcal{U}\mathfrak{g}$ ; the map  $\epsilon$  is the imbedding by  $h \mapsto 1 \otimes h \otimes 1$ . Thus, the image of the map  $\epsilon$  is contained in the center of the algebra  $\mathbf{U}_c$ .

Below, we use the notation introduced at the beginning of §4.2 and observe that any  $\mathbf{U}_c$ -module may be viewed as an  $\mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t}$ -module, via  $\epsilon$ .

- Given  $\lambda \times \nu \in \mathfrak{t}^* \times \mathfrak{t}^*$ , let  $\mathbf{MU}_{\lambda, \nu, c}$  be the full abelian subcategory of  $D^b(\mathbf{U}_c\text{-mod})$  whose objects are  $\mathbf{U}_c$ -modules  $M$  such that one has  $M = M^{(I_{\lambda, \nu})}$ , resp.  $\mathbf{DU}_{\lambda, \nu, c}$  be the full triangulated subcategory of  $D^b(\mathbf{U}_c\text{-mod})$  whose objects are complexes  $M$ , of  $\mathbf{U}_c$ -modules, such that one has  $H^*(M) = [H^*(M)]^{(I_{\lambda, \nu})}$ .

For any object  $M \in D^b(\mathbf{U}_c\text{-mod})$  such that the  $\mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t}$ -action on  $H^*(M)$  is locally finite, there is a canonical direct sum decomposition  $M = \bigoplus_{\lambda \times \nu \in \mathfrak{t}^* \times \mathfrak{t}^*} M^{(\lambda, \nu)}$  such that  $M^{(\lambda, \nu)} \in \mathbf{DU}_{\lambda, \nu, c}$  for all  $\lambda, \nu$ .

Similarly, the composite  $j \circ (\kappa_{x,y} \otimes \text{Id}) \circ \epsilon$ , in diagram (4.4.2), makes any  $\mathcal{D}_{x,y}^c$ -module an  $\mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{t}$ -module. Using this, one shows that for any cosets  $\bar{\lambda}, \bar{\nu} \in \mathfrak{t}^*/\mathfrak{t}_{\mathbb{Z}}^*$  such that  $(\bar{\lambda}, \bar{\nu}) \in \bar{\mathfrak{t}}_{x,y}^\perp$  and any monodromic complex  $\mathcal{V} \in \mathbf{D}\text{mon}_{\bar{\lambda}, \bar{\nu}, c}(\mathfrak{B}_{1,1})$ , there is a canonical ‘derived’ spectral decomposition:

$$R\Gamma(\mathfrak{B}_{x,y}, \mathcal{V}) = \bigoplus_{(\lambda, \nu) \bmod (\mathfrak{t}_{\mathbb{Z}}^* \oplus \mathfrak{t}_{\mathbb{Z}}^*) = (\bar{\lambda}, \bar{\nu})} R\Gamma(\mathfrak{B}_{x,y}, \mathcal{V})^{(\lambda, \nu)} \quad \text{where} \quad R\Gamma(\mathfrak{B}_{x,y}, \mathcal{V})^{(\lambda, \nu)} \in \mathbf{DU}_{\lambda, \nu, c}. \quad (4.4.3)$$

Now, recall the maps  $p, q$  from diagram (4.3.1). The statement of part (i) of the following theorem is a straightforward generalization of a result due to Hotta-Kashiwara, [HK2], Theorem 1.

**Theorem 4.4.4.** *Let  $\lambda \in \mathfrak{t}^*$  be a dominant regular weight, let  $\bar{\lambda}$  be the image of  $\lambda$  under the projection  $\mathfrak{t}^* \rightarrow \mathfrak{t}^*/\mathfrak{t}_{\mathbb{Z}}^*$ , resp.  $\theta$  be the image of  $\lambda$  under the projection  $\mathfrak{t}^* \rightarrow \mathfrak{t}^*/W$ , and  $\Theta$  be the image of  $\theta$  under the projection  $\mathfrak{t}^*/W \rightarrow \mathfrak{t}^*/W_{\text{aff}}$ . Then, for any  $c \neq -1, -2, \dots, 1-n$ , we have*

(i) *The following diagram of functors commutes*

$$\begin{array}{ccc} & \mathbf{Dmon}_{\bar{\lambda}, -\bar{\lambda}, c}(\mathfrak{B}_{1,1}) & \\ \nu \mapsto R\Gamma(\mathfrak{B}, \mathcal{V})^{(\lambda, -\lambda-2\rho)} \swarrow & & \searrow p_* q^![-\dim \mathcal{B}] \\ \mathbf{DU}_{\lambda, -\lambda-2\rho, c} & \xrightarrow{M \mapsto \mathcal{D}_{\mathfrak{X}}^c \otimes_{\mathbf{U}_c}^L M} & \mathbf{D}\mathcal{C}_{\Theta, c}(\mathfrak{X}). \end{array}$$

(ii) *There is an isomorphism of functors  $\mathbf{D}\mathcal{C}_{\Theta, c}(\mathfrak{X}) \rightarrow D^b(\mathbf{U}_c\text{-mod})$ :*

$$[R\Gamma(\mathfrak{B}_{1, w_0}, \text{HC}(-))]^{(\lambda, \lambda^\dagger)} \cong [R\Gamma(\mathfrak{X}, -)]^{(\theta)}.$$

**4.5. Proof of Theorem 4.3.3 and Theorem 4.4.4.** We are going to use one general property of the convolution functor  $(-) * \mathcal{R}_!^{\lambda, \lambda^\dagger}$ , see (3.6.4), that can be deduced from [BB2], Theorem 12.

Let  $x, y \in W$  and  $\nu, \lambda \in \mathfrak{t}^*$ . The property says that, in the bounded derived category of  $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$ -modules, one has a canonical quasi-isomorphism

$$[R\Gamma(\mathfrak{B}_{x, y}, \varpi \otimes \mathcal{V})]^{(\nu, -\lambda-2\rho)} \xrightarrow{\sim} [R\Gamma(\mathfrak{B}_{x, w_0 y}, \mathcal{V} * \mathcal{R}_!^{\lambda, \lambda^\dagger})]^{(\nu, \lambda^\dagger)}, \quad \forall \mathcal{V} \in \mathbf{Dmon}_{\bar{\nu}, -\bar{\lambda}, c}(\mathfrak{B}_{x, y}). \quad (4.5.1)$$

*Proof of Theorem 4.4.4.* The proof of part (i) is identical to the proof of [HK2], Theorem 1, and will be omitted. We now prove part (ii) of Theorem 4.4.4. It will be convenient to introduce the following notation:  $\text{hc}(-) := q_! p^*(-)[\dim \mathcal{B}]$ , resp.  $\text{ch}(-) := p_* q^!(-)[-\dim \mathcal{B}]$ . These functors form an adjoint pair.

We prove first an auxiliary result saying that:

*There is an isomorphism between the following two functors  $\mathbf{Dmon}_{\bar{\lambda}, -\bar{\lambda}, c}(\mathfrak{B}_{1,1}) \rightarrow D^b(\mathbf{U}_c\text{-mod})$ :*

$$[R\Gamma(\mathfrak{B}_{1,1}, \text{hc}(-))]^{(\lambda, -\lambda-2\rho)} \cong [R\Gamma(\mathfrak{X}, -)]^{(\theta)}. \quad (4.5.2)$$

To prove this, observe that our assumptions on  $\lambda$  and  $c$  insure that one has mutually quasi-inverse Beilinson-Bernstein triangulated equivalences

$$\mathbf{DU}_{\lambda, -\lambda-2\rho, c} \xrightleftharpoons[\text{[} R\Gamma(\mathfrak{B}_{1,1}, -) \text{]}^{(\lambda, -\lambda-2\rho)}]{\mathcal{D}_{1,1}^c \otimes_{\mathbf{U}_c}^L (-)} \mathbf{Dmon}_{\bar{\lambda}, -\bar{\lambda}, c}(\mathfrak{B}_{1,1}). \quad (4.5.3)$$

Let  $\mathcal{F} \in \mathbf{D}\mathcal{C}_{\Theta, c}(\mathfrak{X})$  and  $M \in \mathbf{DU}_{\lambda, -\lambda-2\rho, c}$ . Put  $F := R\Gamma(\mathfrak{X}, \mathcal{F})$  and  $\mathcal{M} := \mathcal{D}_{1,1}^c \otimes_{\mathbf{U}_c}^L M \in \mathbf{Dmon}_{\bar{\lambda}, -\bar{\lambda}, c}(\mathfrak{B}_{1,1})$ . We compute

$$\begin{aligned} R\text{Hom}_{\mathbf{U}_c}(M, [R\Gamma(\mathfrak{B}_{1,1}, \text{hc}(\mathcal{F}))]^{(\lambda, -\lambda-2\rho)}) &= R\text{Hom}_{\mathcal{D}_{1,1}^c}(\mathcal{M}, \text{hc}(\mathcal{F})) && \text{by (4.5.3)} \\ &= R\text{Hom}_{\mathcal{D}_{\mathfrak{X}}^c}(\text{ch}(\mathcal{M}), \mathcal{F}) && \text{by adjunction} \\ &= R\text{Hom}_{\mathcal{D}_{\mathfrak{X}}^c}(\mathcal{D}_{\mathfrak{X}}^c \otimes_{\mathbf{U}_c}^L M^{(\lambda, -\lambda-2\rho)}, \mathcal{F}) && \text{by part (i)} \\ &= R\text{Hom}_{\mathbf{U}_c}(M^{(\lambda, -\lambda-2\rho)}, \mathcal{F}) \\ &= R\text{Hom}_{\mathbf{U}_c}(M^{(\lambda, -\lambda-2\rho)}, F) \\ &= R\text{Hom}_{\mathbf{U}_c}(M, F) = R\text{Hom}_{\mathbf{U}_c}(M, F^{(\theta)}), \end{aligned}$$

where the last two equalities hold since  $M = M^{(\lambda, -\lambda-2\rho)}$ .

Thus, we have established, for any  $\mathcal{F} \in \mathbf{D}\mathcal{C}_{\Theta,c}(\mathfrak{X})$ ,  $M \in \mathbf{D}\mathbf{U}_{\lambda,-\lambda-2\rho,c}$ , a functorial isomorphism:

$$R\mathrm{Hom}_{\mathbf{U}_c}(M, R\Gamma(\mathfrak{B}_{1,1}, \mathrm{hc}(\mathcal{F}))^{(\lambda)}) = R\mathrm{Hom}_{\mathbf{U}_c}(M, R\Gamma(\mathfrak{X}, \mathcal{F})^{(\theta)}).$$

Such an isomorphism clearly yields an isomorphism of functors claimed in (4.5.2).

To complete the poof of part (ii) of Theorem 4.4.4(ii), we combine (4.5.2) with the quasi-isomorphism (4.5.1) for  $\mathcal{V} := \mathrm{hc}(\mathcal{F})$ . This way, using the definition of the functor  $\mathrm{HC}$ , we obtain the isomorphism of functors stated in part (ii) of the theorem.  $\square$

*Proof of Theorem 4.3.3.* We have that both  $\lambda$  and  $\lambda^\dagger$  are dominant regular weights and  $c \neq -1, -2, \dots, 1-n$ . Therefore, according to the Beilinson-Bernstein theorem, each of the two functors  $\Gamma(\mathfrak{B}_{1,w_0}, -)^{(\lambda, \lambda^\dagger)} : \mathbf{Mon}_{\bar{\lambda}, \bar{\lambda}^\dagger, c} \rightarrow \mathbf{MU}_{\lambda, \lambda^\dagger, c}$  and  $\Gamma(\mathfrak{X}, -) : \mathcal{D}_{\mathfrak{X}}^c\text{-mod} \rightarrow \mathcal{D}(\mathfrak{X}, c)\text{-mod}$  is exact and yields an equivalence of *abelian* categories. Also, the functor  $F \mapsto F^{(\theta)}$  is clearly exact on the category of  $\mathfrak{Z}$ -locally finite modules.

Thus, the isomorphism of functors in Theorem 4.4.4(ii) implies that

$$[\Gamma(\mathfrak{B}_{1,w_0}, \mathrm{HC}(-))]^{(\lambda, \lambda^\dagger)} : \mathcal{C}_{\Theta,c}(\mathfrak{X}) \rightarrow \mathbf{D}\mathbf{U}_{\lambda, \lambda^\dagger, c}$$

is an exact functor. Furthermore, this functor is a composite of the functor  $\mathrm{HC}$  and the functor  $[\Gamma(\mathfrak{B}_{1,w_0}, -)]^{(\lambda, \lambda^\dagger)}$ , which is an equivalence of the corresponding abelian categories.

It follows that  $\mathrm{HC}$  must itself induce an exact functor between abelian categories.  $\square$

## 5. MIRABOLIC HARISH-CHANDRA $\mathcal{D}$ -MODULE

5.1. There is an especially important family of mirabolic character  $\mathcal{D}$ -modules that was introduced in [GG, § 7.4]. To define these  $\mathcal{D}$ -modules, recall the map (4.1.1) and the subsets  $\mathfrak{Z}_\theta^c = v\zeta(\mathfrak{Z}_\theta) \subset \mathcal{D}(\mathfrak{X}, c)$ , see §4.2.

**Definition 5.1.1.** For any  $(\theta, c) \in (\mathfrak{t}^*/W) \times \mathbb{C}$ , we define a  $\mathcal{D}_{\mathfrak{X}}^c$ -module called *Harish-Chandra  $\mathcal{D}_{\mathfrak{X}}^c$ -module*, resp. a projective limit of  $\mathcal{D}_{\mathfrak{X}}^c$ -modules called *generalized Harish-Chandra  $\mathcal{D}$ -module*, as follows

$$\mathcal{G}^{\theta,c} := \mathcal{D}_{\mathfrak{X}}^c / (\mathcal{D}_{\mathfrak{X}}^c \mathfrak{g} + \mathcal{D}_{\mathfrak{X}}^c \mathfrak{Z}_\theta^c), \quad \text{resp.} \quad \widehat{\mathcal{G}}^{\theta,c} := \lim_{m \rightarrow \infty} \mathrm{proj} \mathcal{D}_{\mathfrak{X}}^c / (\mathcal{D}_{\mathfrak{X}}^c \mathfrak{g} + \mathcal{D}_{\mathfrak{X}}^c (\mathfrak{Z}_\theta^c)^m).$$

It is clear that we have  $\mathcal{G}^{\theta,c} \in \mathcal{C}_{\Theta,c}$ , where  $\Theta \in \mathfrak{t}^*/W_{\mathrm{aff}}$  is the image of  $\theta$  under the projection  $\mathfrak{t}^*/W \rightarrow \mathfrak{t}^*/W_{\mathrm{aff}}$ . Therefore, Proposition 4.3.2 implies that  $\mathcal{G}^{\theta,c}$  is a regular holonomic  $\mathcal{D}$ -module.

Our goal is to provide a geometric construction of the Harish-Chandra  $\mathcal{D}$ -module  $\mathcal{G}^{\theta,c}$  similar to one given in [HK1] in the classical case.

To this end, let  $U$  be the unique Zariski open and dense  $G \times \mathbb{T} \times \mathbb{T}$ -orbit in  $\mathfrak{B}_{1,w_0}$ . One may mimic definitions of the local systems  $\mathcal{J}^\lambda$  and  $\widehat{\mathcal{J}}^\lambda$ , on  $\mathcal{U}$ , see §3.6, and introduce analogous local systems, more precisely, monodromic  $\mathcal{D}_U^c$ -modules  $\mathcal{L}^{\lambda,c}$  and  $\widehat{\mathcal{L}}^{\lambda,c}$ .

Write  $j : U \hookrightarrow \mathfrak{B}_{1,w_0}$  for the open imbedding.

**Theorem 5.1.2.** *Let  $\lambda \in \mathfrak{t}^*$  be a sufficiently dominant regular weight, and let  $\theta$  be the image of  $\lambda$  under the projection  $\mathfrak{t}^* \rightarrow \mathfrak{t}^*/W$ . Then, for sufficiently large real  $c \gg 0$ , in  $\mathbf{D}\mathcal{C}_c(\mathfrak{X})$  there is an isomorphism*

$$\mathcal{G}^{\theta,c} \cong \mathrm{CH}(j_! \mathcal{L}^{\lambda,c}), \quad \text{resp.} \quad \widehat{\mathcal{G}}^{\theta,c} \cong \mathrm{CH}(j_! \widehat{\mathcal{L}}^{\lambda,c}). \quad (5.1.3)$$

The theorem provides a purely geometric construction of the perverse sheaf that corresponds to the Harish-Chandra  $\mathcal{D}$ -module  $\mathcal{G}^{\theta,c}$  via the Riemann-Hilbert correspondence.

The proof of Theorem 5.1.2 will occupy sections 5.2-5.6.

**Corollary 5.1.4.** *With the assumptions of Theorem 4.3.3, we have that  $\widehat{\mathcal{G}}^{\theta,c}$  is a projective (pro)-object of the category  $\mathcal{C}_c(\mathfrak{X})$ .*



*Proof of Corollary.* In general, let  $P$  be a projective (pro)-object of the abelian category  $\mathbf{Mon}_c(\mathfrak{B}_{1,w_0})$  such that the complex  $\mathbf{CH}(P)$  is concentrated in degree zero.

We claim that  $\mathrm{Hom}_{\mathcal{D}_{\mathfrak{X}}^c}(\mathbf{CH}(P), -)$  is an exact functor on the category  $\mathcal{C}_c(\mathfrak{X})$ . To check this, we use adjunction and obtain

$$\mathrm{Hom}_{\mathcal{D}_{\mathfrak{X}}^c}(\mathbf{CH}(P), \mathcal{M}) = \mathrm{Hom}_{\mathcal{D}_{\mathfrak{B}}^c}(P, \mathbf{HC}(\mathcal{M})), \quad \forall \mathcal{M} \in \mathcal{C}_c(\mathfrak{X}).$$

Here, the functor  $\mathcal{M} \mapsto \mathbf{HC}(\mathcal{M})$  is exact by Theorem 4.3.3, and the functor  $\mathrm{Hom}_{\mathcal{D}_{\mathfrak{B}}^c}(P, -)$  is exact since  $P$  is projective. Thus, we have proved our claim. We conclude that  $\mathbf{CH}(P)$  is a projective (pro)-object of the category  $\mathcal{C}_c(\mathfrak{X})$ .

To complete the proof of the corollary, we observe that  $\widehat{\mathcal{L}}^{\lambda,c}$  is a projective (pro)-object in the category of  $G$ -equivariant local systems on  $U$ , the open  $G \times \mathbb{T} \times \mathbb{T}$ -orbit. Therefore,  $j_! \widehat{\mathcal{L}}^{\lambda,c}$  is a projective (pro)-object of the category  $\mathbf{Mon}_c(\mathfrak{B}_{1,w_0})$ . The result follows.  $\square$

5.2. We begin with a general setting.

Let  $X$  be a smooth variety and let  $\mathcal{D}$  be a TDO on  $X$ . Let  $\mathfrak{G}$  be an  $m$ -dimensional filtered Lie algebra equipped with a filtration preserving Lie algebra morphism  $\mathfrak{G} \rightarrow \mathcal{D}$ ,  $u \mapsto \vec{u}$ , into (not necessarily first order) twisted differential operators. Let  $\mathcal{D}\mathfrak{G}$  be the left ideal in  $\mathcal{D}$  generated by the image of  $\mathfrak{G}$ .

We recall the following version of a result due to G. Schwarz [Sc], §8, and M. Holland, [H], Proposition 2.4, independently.

**Lemma 5.2.1.** *Assume that the image of  $\mathrm{gr} \mathfrak{G} \rightarrow \mathrm{gr} \mathcal{D}$ , the associated graded morphism, gives a regular sequence in  $\mathrm{gr} \mathcal{D}$ . Then, the natural map*

$$\mathrm{gr} \mathcal{D} / \mathrm{gr} \mathcal{D} \mathrm{gr} \mathfrak{G} \rightarrow \mathrm{gr}(\mathcal{D} / \mathcal{D}\mathfrak{G}), \quad (5.2.2)$$

*is a bijection. Moreover, the standard Chevalley-Eilenberg complex associated with the action of the Lie algebra  $\mathfrak{G}$  on  $\mathcal{D}$  by right multiplication,*

$$0 \rightarrow \mathcal{D} \otimes \Lambda^m \mathfrak{G} \rightarrow \mathcal{D} \otimes \Lambda^{m-1} \mathfrak{G} \rightarrow \dots \rightarrow \mathcal{D} \otimes \Lambda^2 \mathfrak{G} \rightarrow \mathcal{D} \otimes \mathfrak{G} \rightarrow \mathcal{D} \rightarrow 0, \quad (5.2.3)$$

*is a free  $\mathcal{D}$ -module resolution of  $\mathcal{D} / \mathcal{D}\mathfrak{G}$ , a left  $\mathcal{D}$ -module.*

The algebra morphism  $\mathrm{gr} \mathfrak{G} \rightarrow \mathrm{gr} \mathcal{D}$  is induced by a moment map  $\mu : T^*X \rightarrow (\mathrm{gr} \mathfrak{G})^*$ . The condition of the lemma that the image of  $\mathrm{gr} \mathfrak{G}$  be a regular sequence may be reformulated as a requirement that the Koszul complex

$$0 \rightarrow \mathcal{O}_{T^*X} \otimes \Lambda^m(\mathrm{gr} \mathfrak{G}) \rightarrow \mathcal{O}_{T^*X} \otimes \Lambda^{m-1}(\mathrm{gr} \mathfrak{G}) \rightarrow \dots \rightarrow \mathcal{O}_{T^*X} \otimes \Lambda^1(\mathrm{gr} \mathfrak{G}) \rightarrow \mathcal{O}_{T^*X} \rightarrow 0, \quad (5.2.4)$$

be a resolution of the structure sheaf  $\mathcal{O}_{\mu^{-1}(0)}$ , by free  $\mathcal{O}_{T^*X}$ -modules. The latter condition is also equivalent to the condition that  $\mu^{-1}(0)$ , the scheme-theoretic zero fiber of the map  $\mu$ , be a complete intersection.

5.3. Let  $\lambda, \nu \in \mathfrak{t}^*$ . We may view the pair  $(\lambda, \nu)$  as a linear function  $\lambda \times \nu : \mathfrak{t} \oplus \mathfrak{t} \rightarrow \mathbb{C}$  and write  $\mathfrak{t}^{\lambda, \nu} \subset \mathcal{D}_{x,y}^c$  for the subspace associated with the  $\mathbb{T} \times \mathbb{T}$ -action on  $\mathfrak{B}_{x,y}$  and with the character  $\lambda \times \nu$ , as explained in §2.3. Let  $\mathcal{D}_{x,y}^c \mathfrak{t}^{\lambda, \nu} \subset \mathcal{D}_{x,y}^c$  be the corresponding left ideal.

We observe that one actually has an equality  $\mathcal{D}_{x,y}^c \mathfrak{t}^{\lambda, \nu} = \mathcal{D}_{x,y}^c$  unless the linear function  $\lambda \times \nu$  annihilates the subspace  $\mathfrak{t}_{x,y} \subset \mathfrak{t} \oplus \mathfrak{t}$ , i.e. unless we have  $\lambda \times \nu \in \mathfrak{t}_{x,y}^\perp \subset \mathfrak{t}^* \oplus \mathfrak{t}^*$ . It is clear that the latter holds iff there exists an element  $\gamma \in \mathfrak{t}^*$  such that  $\lambda = x(\gamma)$  and  $\nu = -y(\gamma)$ .

For any  $c \in \mathbb{C}$  and  $x, y \in W$ , we introduce a family of left  $\mathcal{D}_{x,y}^c$ -modules

$$\mathcal{V}_{x,y}^{\lambda, \nu, c} := \mathcal{D}_{x,y}^c / (\mathcal{D}_{x,y}^c \mathfrak{g} + \mathcal{D}_{x,y}^c \mathfrak{t}^{\lambda, \nu}) \in \mathbf{Mon}_c(\mathfrak{B}_{x,y}), \quad \lambda \times \nu \in \mathfrak{t}_{x,y}^\perp. \quad (5.3.1)$$

It is clear that  $\mathcal{V}_{x,y}^{\lambda, \nu, c}$  is a  $G$ -equivariant regular holonomic  $\mathcal{D}$ -module, by Corollary 3.3.2(i).

Next, put  $D_{x,y} := \Gamma(\mathfrak{B}_{x,y}, \mathcal{D}_{x,y}^c)$ .

**Lemma 5.3.2.** *For any  $(\lambda, \nu) \in \mathfrak{t}_{x,y}^\perp$ , one has:  $R\Gamma(\mathfrak{B}_{x,y}, \mathcal{V}_{x,y}^{\lambda,\nu,c}) \simeq D_{x,y}/(D_{x,y} \mathfrak{g} + D_{x,y} \mathfrak{t}^{\lambda,\nu})$ .*

*Proof.* We use the Koszul complex (5.2.3) in the case where  $X = \mathfrak{B}_{x,y}$ . The Koszul complex is acyclic, by Proposition 3.3.1(ii) and Lemma 5.2.1. It is well known that one has  $R^\ell \Gamma(\mathfrak{B}_{x,y}, \mathcal{D}_{x,y}^c) = 0$ , for any  $\ell > 0$ , since this is the case for the associated graded algebra, thanks to the Grauert-Riemenschneider theorem. Thus, the Koszul complex (5.2.3) provides a *left*  $\Gamma$ -acyclic resolution of  $\mathcal{V}_{x,y}^{\lambda,\nu,c}$ . Computing  $R\Gamma(\mathfrak{B}_{x,y}, \mathcal{V}_{x,y}^{\lambda,\nu,c})$  via this resolution yields the result.  $\square$

**Corollary 5.3.3.** *Assume  $\lambda \in \mathfrak{t}^*$  is regular, and  $c \neq -1, -2, \dots, 1 - n$ . We have isomorphisms:*

$$\mathcal{V}_{1,1}^{\lambda,-\lambda,c} \simeq \mathcal{V}_{1,w_0}^{\lambda,\lambda^\dagger,c} * \mathcal{R}_*^{-\lambda^\dagger,-\lambda}, \quad \text{resp.} \quad \mathcal{V}_{1,1}^{-\lambda,\lambda,-c} \simeq \mathcal{V}_{1,w_0}^{-\lambda,-\lambda^\dagger,-c} * \mathcal{R}_!^{\lambda^\dagger,\lambda}.$$

*Proof.* We only prove the second isomorphism, the proof of the other one being similar. To do this, we first verify, by a simple direct calculation that one has a natural isomorphism of  $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$ -modules

$$D_{1,1}^{-c}/(D_{1,1}^{-c} \mathfrak{g} + D_{1,1}^{-c} \mathfrak{t}^{-\lambda,\lambda}) \cong D_{1,w_0}^{-c}/(D_{1,w_0}^{-c} \mathfrak{g} + D_{1,w_0}^{-c} \mathfrak{t}^{-\lambda,-\lambda^\dagger}).$$

Hence, using Lemma 5.3.2 we may rewrite the above isomorphism as follows

$$[R\Gamma(\mathfrak{B}_{1,1}, \mathcal{V}_{1,1}^{-\lambda,\lambda,-c})]^{(-\lambda,\lambda)} \cong [R\Gamma(\mathfrak{B}_{1,w_0}, \mathcal{V}_{1,w_0}^{-\lambda,-\lambda^\dagger,-c})]^{(-\lambda,-\lambda^\dagger)}.$$

Further, by (4.5.1), the object on the right hand side above is isomorphic, in the bounded derived category of  $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$ -modules, to  $[R\Gamma(\mathfrak{B}_{1,w_0}, \mathcal{V}_{1,w_0}^{-\lambda,-\lambda^\dagger,-c} * \mathcal{R}_!^{\lambda^\dagger,\lambda})]^{(-\lambda,\lambda)}$ .

Therefore, the functor  $[R\Gamma(\mathfrak{B}_{1,1}, -)]^{(-\lambda,\lambda)}$  takes  $\mathcal{D}$ -modules  $\mathcal{V}_{1,1}^{-\lambda,\lambda,-c}$  and  $\mathcal{V}_{1,w_0}^{-\lambda,-\lambda^\dagger,-c} * \mathcal{R}_!^{\lambda^\dagger,\lambda}$  to isomorphic objects. The result now follows from the Beilinson-Bernstein theorem.  $\square$

5.4. We keep the setting of section 3.2 and let the Weyl group act on  $\mathfrak{t}^*$  via the dot-action. We use the identification  $\text{Spec } \mathfrak{Z} \cong \mathfrak{t}^*/W$ , provided by the Harish-Chandra isomorphism.

The canonical isomorphism  $\tau : \mathcal{U}\mathfrak{g} \rightarrow (\mathcal{U}\mathfrak{g})^{\text{op}}$  restricts to an automorphism  $\tau : \mathfrak{Z} \rightarrow \mathfrak{Z}$ . Write  $\tau : \text{Spec } \mathfrak{Z} \rightarrow \text{Spec } \mathfrak{Z}$  for the induced automorphism. In terms of the Harish-Chandra isomorphism  $\text{Spec } \mathfrak{Z} \cong \mathfrak{t}^*/W$ , one can write  $\tau(W \cdot \lambda) = W \cdot (-\lambda - 2\rho)$ .

Recall the notation from Diagram (4.3.1).

**Proposition 5.4.1.** *Assume  $\lambda \in \mathfrak{t}^*$  is regular, and  $c \neq -1, -2, \dots, 1 - n$ . Write  $\theta \in \mathfrak{t}^*/W$  for the image of  $\lambda$ . Then, we have an isomorphism*

$$\mathcal{G}^{\theta,c} = p_* q^! (\mathcal{V}_{1,1}^{\lambda,-\lambda,c})[-\dim \mathcal{B}]. \quad (5.4.2)$$

*Proof.* Given  $\theta \in \text{Spec } \mathfrak{Z}$ , we introduce the notation  $\mathcal{U}_\theta := \mathcal{U}\mathfrak{g}/\mathcal{U}\mathfrak{g} \cdot \mathfrak{Z}_\theta$ . Further, write  $\mathcal{U}_{\theta,\tau(\theta),c} := \mathcal{U}_\theta \otimes \mathcal{U}_{\tau(\theta)} \otimes \mathcal{U}_c$ . We introduce a coproduct  $\Delta$ , an algebra map defined on generators as follows

$$\Delta : \mathcal{U}\mathfrak{g} \longrightarrow (\mathcal{U}\mathfrak{g})^{\otimes 3} \twoheadrightarrow \mathcal{U}_{\theta,\tau(\theta),c}, \quad x \mapsto 1 \otimes 1 \otimes x + 1 \otimes x \otimes 1 + x \otimes 1 \otimes 1.$$

Clearly, we have  $\mathcal{G}^{\theta,c} = \mathcal{D}_{G,\theta} \otimes_{\mathcal{U}_{\theta,\tau(\theta),c}} (\mathcal{U}_{\theta,\tau(\theta),c}/\mathcal{U}_{\theta,\tau(\theta),c} \Delta(\mathfrak{g}))$ , where  $\mathcal{D}_{G,\theta} := \mathcal{D}_G/\mathcal{D}_G \text{lr}(\mathfrak{Z}_\theta \otimes 1)$ . Observe further that the  $\mathcal{D}$ -module  $\mathcal{G}^{\theta,c}$  is the top (zeroth) cohomology module of the complex

$${}^L \mathcal{G}^{\theta,c} := \mathcal{D}_{G,\theta} \otimes_{\mathcal{U}_{\theta,\tau(\theta),c}} {}^L (\mathcal{U}_{\theta,\tau(\theta),c}/\mathcal{U}_{\theta,\tau(\theta),c} \Delta(\mathfrak{g})). \quad (5.4.3)$$

To complete the proof, we must show, in view of Theorem 4.4.4(i), that the canonical morphism  ${}^L \mathcal{G}^{\theta,c} \rightarrow \mathcal{G}^{\theta,c}$  is a quasi-isomorphism.

To this end, recall that the canonical line bundle  $\omega_{\mathcal{B}}$  is isomorphic to the line bundle  $\mathcal{O}_{\mathcal{B}}(-2\rho)$ . Thus, we must prove  $R\Gamma(\mathcal{B} \times \mathcal{B} \times \mathbb{P}, \mathcal{V}_{1,1}^{\lambda,-\lambda-2\rho,c}) = \mathcal{U}_{\theta,\tau(\theta),c}/\mathcal{U}_{\theta,\tau(\theta),c} \Delta(\mathfrak{g})$ . But this last isomorphism holds by Lemma 5.3.2, and we are done.  $\square$

5.5. We are going to reduce the proof of Theorem 5.1.2 to Theorem 2.3.1 of Section 2.

To this end, recall that associated with any integral weight  $\nu \in \mathbb{T}_{\mathbb{Z}}^*$ , there is a  $G$ -equivariant line bundle  $\mathcal{O}(\nu)$  on  $\mathcal{B}$ . We put  $\mathcal{L} := \mathcal{O}_{\tilde{\mathcal{B}}_{1,w_0}} \boxtimes \mathcal{O}_{\mathbb{P}}(n)$ , a  $G \times \mathbb{T} \times \mathbb{T}$ -equivariant line bundle on  $\mathfrak{B}_{1,w_0}$ .

Further, we introduce a character  $\phi := (0, 2\rho, 2\rho) \in \mathfrak{g}^* \times \mathfrak{t}_{1,w_0}^\perp \subset \mathfrak{g}^* \times \mathfrak{t}^* \times \mathfrak{t}^*$ .

The following result shows, in particular, that the open imbedding  $j : U \hookrightarrow \mathfrak{B}_{1,w_0}$  is an affine morphism.

**Lemma 5.5.1.** *There is a  $\phi$ -semi-invariant (with respect to the  $G \times \mathbb{T} \times \mathbb{T}$ -action) section  $s \in \Gamma(\mathfrak{B}_{1,w_0}, \mathcal{L})$ , such that one has  $s^{-1}(0) = \mathfrak{B}_{1,w_0} \setminus U$ .*

*Proof.* Let  $V = \mathbb{C}^n$  and let  $\omega_i$ ,  $i = 1, \dots, n-1$ , stand for the fundamental weights of  $G$ . We will write  $\omega_0 = 0$ . Fix a volume functional  $\text{vol} : \Lambda^n V \rightarrow \mathbb{C}$ .

The line bundle  $\mathcal{O}(\omega_1)$  on  $\mathcal{B}$  descends to the line bundle  $\mathcal{O}_{\mathbb{P}}(1)$  on  $\mathbb{P}$ , and the space of its global sections is canonically isomorphic to  $V^*$ . Let  $s_i$  stand for the global section of  $\mathcal{O}(\omega_i) \boxtimes \mathcal{O}(\omega_{n-i})$  which sends  $v_i \otimes v_{n-i} \in \Lambda^i V \otimes \Lambda^{n-i} V = \Gamma[\mathcal{B} \times \mathcal{B}, \mathcal{O}(\omega_i) \boxtimes \mathcal{O}(\omega_{n-i})]^*$  to the volume of  $v_i \wedge v_{n-i}$ . Let  $s_i$  be its lift to a global section of  $\mathcal{O}(\omega_i) \boxtimes \mathcal{O}(\omega_{n-i}) \boxtimes \mathcal{O}_{\mathbb{P}}$ .

Let  $\varsigma_j$ ,  $0 \leq j \leq n-1$ , be the global section of  $\mathcal{O}(\omega_j) \boxtimes \mathcal{O}(\omega_{n-1-j}) \boxtimes \mathcal{O}_{\mathbb{P}}(1)$  such that, for any

$$v_j \otimes v_{n-1-j} \otimes v \in \Lambda^j V \otimes \Lambda^{n-1-j} V \otimes V = \Gamma(\mathcal{B} \times \mathcal{B} \times \mathbb{P}, \mathcal{O}(\omega_j) \boxtimes \mathcal{O}(\omega_{n-1-j}) \boxtimes \mathcal{O}_{\mathbb{P}}(1))^*,$$

we have that  $\langle \varsigma_j, v_j \otimes v_{n-1-j} \otimes v \rangle = \text{vol}(v_j \wedge v_{n-1-j} \wedge v)$ .

Finally, we denote by  $s$  the global section  $s_1 \dots s_{n-1} \varsigma_0 \dots \varsigma_{n-1}$  of the product  $\mathcal{O}(2\rho) \boxtimes \mathcal{O}(2\rho) \boxtimes \mathcal{O}(n) \simeq \omega_{\mathcal{B} \times \mathcal{B} \times \mathbb{P}}^{-1}$  of the above line bundles on  $\mathcal{B} \times \mathcal{B} \times \mathbb{P}$ .

Now, we use an explicit classification of  $G$ -diagonal orbits in  $\mathcal{B} \times \mathcal{B} \times \mathbb{P}$  given in [MWZ, §2.11]. The classification shows that any codimension one  $G$ -orbit in  $\mathfrak{B}_{1,w_0}$  is equal (locally) to the zero locus of either one of the sections  $s_i$  or of one of the sections  $\varsigma_j$ . One deduces that the set  $(\mathcal{B} \times \mathcal{B} \times \mathbb{P}) \setminus s^{-1}(0)$  is the open  $G$ -diagonal orbit in  $\mathcal{B} \times \mathcal{B} \times \mathbb{P}$ . The statement of the lemma easily follows from this.  $\square$

*Remark 5.5.2.* It is likely that there is an explicit closed expression, as a product of linear factors, for the  $b$ -function associated with the section  $s$  of Lemma 5.5.1. We expect that such an expression may be obtained by adapting arguments used by M. Kashiwara in [K3]. The knowledge of the roots of the  $b$ -function would make it possible to give explicit sharp bounds on the parameter ‘ $c$ ’ which are necessary and sufficient for the statement of Theorem 5.1.2 to hold true.

**Lemma 5.5.3.** *Let  $\mathcal{L}$  and  $\phi$  be as in Lemma 5.5.1. For any  $x \in \mathfrak{B}_{1,w_0} \setminus U$ , using the notation of formula (2.3.2), we have  $\chi_{\mathcal{L},x} \neq \phi|_{\mathfrak{g}_x}$ .*

*Proof.* Fix a point  $x \in \mathfrak{B}_{1,w_0} \setminus U$ , and let  $G_x$  be its isotropy group in  $G$ . Travkin has shown in [T, Lemma 7] that there exists a 1-parameter subgroup  $\gamma : \mathbb{C}^\times \rightarrow G_x$  and an integer  $m > 0$  such that, for any  $z \in \mathbb{C}^\times$ , one has  $\chi_{\mathcal{L},x} \circ \gamma(z) = z^m$ . On the other hand, since  $\phi := (0, 2\rho, 2\rho)$ , we have  $\phi \circ \gamma(z) = 1$ . We conclude that  $\chi_{\mathcal{L},x} \circ \gamma \neq \phi \circ \gamma$ , hence  $\chi_{\mathcal{L},x} \neq \phi|_{\mathfrak{g}_x}$ .  $\square$

**5.6. Proof of Theorem 5.1.2.** For any  $\lambda \in \mathfrak{t}^*$ , the sheaf  $\mathcal{L}^{\lambda,c} := j^* \mathcal{V}_{1,w_0}^{\lambda, \lambda^\dagger, c}$  is a locally free rank one  $G$ -equivariant  $\mathcal{O}_U$ -module, i.e. a line bundle on  $U$ .

**Lemma 5.6.1.** *For dominant enough  $\lambda$ , and  $c \gg 0$ , the canonical morphisms below induce isomorphisms*

$$\mathcal{V}_{1,w_0}^{\lambda, \lambda^\dagger, c} \xrightarrow{\sim} j_* j^* \mathcal{V}_{1,w_0}^{\lambda, \lambda^\dagger, c}, \quad \text{resp.} \quad j! j^! \mathcal{V}_{1,w_0}^{\lambda, \lambda^\dagger, c} \xrightarrow{\sim} \mathcal{V}_{1,w_0}^{\lambda, \lambda^\dagger, c}.$$

*Proof.* We apply Theorem 2.3.1 to  $X := \mathfrak{B}_{1,w_0}$  and to the section  $s$  from Lemma 5.5.1. Condition (2.3.2) of Theorem 2.3.1 holds thanks to Lemma 5.5.3. Thus, Theorem 2.3.1 says that, given  $\lambda \in \mathfrak{t}^*$ ,  $c \in \mathbb{C}$ , for all integers  $k \gg 0$ , one has

$$\mathcal{V}_{1,w_0}^{-\lambda-k\rho, -\lambda-k\rho, -c-n} \simeq j_* \mathcal{L}_{1,w_0}^{-\lambda-k\rho, -\lambda-k\rho, -c-kn}, \quad \text{resp.} \quad \mathcal{V}_{1,w_0}^{\lambda, \lambda^\dagger+k\rho, c+kn} \simeq j! \mathcal{L}_{1,w_0}^{\lambda, \lambda^\dagger+k\rho, c+kn}.$$

This proves the lemma.  $\square$

We can now complete the proof of Theorem 5.1.2. To this end, we combine Corollary 5.3.3 with Lemma 5.6.1. We deduce that, for all sufficiently dominant enough  $\lambda \in \mathfrak{t}^*$  and  $c \gg 0$ , one has

$$\mathcal{V}_{1,1}^{-\lambda,\lambda,-c} \simeq (j_* \mathcal{L}_{1,w_0}^{-\lambda,-\lambda^\dagger,-c}) * \mathcal{R}_!^{\lambda^\dagger,\lambda}, \quad \text{resp.} \quad \mathcal{V}_{1,1}^{\lambda,-\lambda,c} \simeq (j_* \mathcal{L}_{1,w_0}^{\lambda,\lambda^\dagger,c}) * \mathcal{R}_*^{-\lambda^\dagger,-\lambda}.$$

Thus, using the isomorphism in (5.4.2), we obtain

$$\mathcal{G}^{\theta,c} = p_* q^! (\mathcal{V}_{1,1}^{\lambda,-\lambda,c}) [-\dim \mathcal{B}] = p_* q^! (\varpi \otimes (j_* \mathcal{L}_{1,w_0}^{-\lambda,-\lambda^\dagger,-c} * \mathcal{R}_!^{\lambda^\dagger,\lambda})) = \text{CH}(j_* \mathcal{L}_{1,w_0}^{-\lambda,-\lambda^\dagger,-c}),$$

and the theorem is proved.  $\square$

## 6. FURTHER PROPERTIES OF THE MIRABOLIC HARISH-CHANDRA $D$ -MODULE

6.1. Let  $X$  be a smooth manifold and let  $\Delta : X \rightarrow X \times X$  be the diagonal embedding. Given a class  $\chi \in H^2(X, \Omega_X^{1,2})$ , we put  $\mathcal{D}_{\chi,-\chi} := \mathcal{D}_\chi \boxtimes \mathcal{D}_{-\chi}$ , a TDO on  $X \times X$ . In the notation of [K2], 2.8, we have that  $\Delta^\# \mathcal{D}_{\chi,-\chi}$  is the sheaf of *non-twisted* differential operators on  $X$ . The category  $\mathcal{D}_\chi\text{-mod-}\mathcal{D}_\chi$ , of  $\mathcal{D}_\chi$ - $\mathcal{D}_\chi$ -bimodules, is equivalent to the category of  $\mathcal{D}_\chi \boxtimes \mathcal{D}_{-\chi}$ -modules.

The pushforward  $\mathcal{D}_{\chi,-\chi}$ -module,  $\Delta_* \mathcal{O}_X$ , viewed as a left  $\mathcal{D}_\chi$ -module, is canonically isomorphic to  $\mathcal{D}_\chi \otimes_{\mathcal{O}_X} \omega_X^{-1}$ , see [K2], 2.11. On the other hand,  $\mathcal{D}_\chi$ , the diagonal  $\mathcal{D}_\chi$ - $\mathcal{D}_\chi$ -bimodule viewed as a left  $\mathcal{D}_\chi \boxtimes \mathcal{D}_{-\chi}$ -module, is canonically isomorphic to  $\Delta_* \mathcal{O}_X \otimes \text{pr}_2^* \omega_X$  where  $\text{pr}_2 : X \times X \rightarrow X$  is the second projection.

Given a holonomic  $\mathcal{D}_\chi$ -module  $\mathcal{F}$ , the complex  $R\mathcal{H}om_{\mathcal{D}_\chi}(\mathcal{F}, \mathcal{D}_\chi)$  has the only cohomology in degree  $d = \dim X$ , so it is quasi-isomorphic to  $\mathcal{E}xt_{\mathcal{D}_\chi}^d(\mathcal{F}, \mathcal{D}_\chi)$ . The right action of  $\mathcal{D}_\chi$  on itself gives rise to the right action of  $\mathcal{D}_\chi$  on  $\mathcal{E}xt_{\mathcal{D}_\chi}^d(\mathcal{F}, \mathcal{D}_\chi)$ . Thus we have a  $\mathcal{D}_\chi^{\text{op}} = \mathcal{D}_{-\chi}$ -module  $\mathcal{E}xt_{\mathcal{D}_\chi}^d(\mathcal{F}, \mathcal{D}_\chi)$ , and we define the  $\mathcal{D}_{-\chi}$ -module  $\mathbb{D}(\mathcal{F}) := \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{E}xt_{\mathcal{D}_\chi}^d(\mathcal{F}, \mathcal{D}_\chi)$ .

Fix a Lie algebra  $\mathfrak{g}$  of dimension  $m$ , with *modular* character  $\delta(-) = \text{Tr ad}$ . We use the notation of § 2.3, and write  $\mu : T^*X \rightarrow \mathfrak{g}^*$  for the moment map.

**Lemma 6.1.1.** *Assume that the moment map  $\mu : T^*X \rightarrow \mathfrak{g}^*$  is flat. Then, we have a natural isomorphism*

$$R\mathcal{H}om_{\mathcal{D}}(\mathcal{D}/\mathcal{D}\mathfrak{g}^\psi, \mathcal{D}) \cong \mathcal{D}^{\text{op}}/\mathcal{D}^{\text{op}}\mathfrak{g}^{\delta-\psi}[-\dim X].$$

*Proof.* By assumptions, we may apply Lemma 5.2.1 to the Lie algebra  $\mathfrak{G} := \mathfrak{g}$ , all placed in filtration degree 1, and to the Lie algebra map  $\mathfrak{g} \rightarrow \mathcal{D}$ ,  $u \mapsto \vec{u} - \psi(u) \cdot 1$ . We conclude that the corresponding complex (5.2.3) provides a free  $\mathcal{D}$ -module resolution of  $\mathcal{D}/\mathcal{D}\mathfrak{g}^\psi$ .

Clearly, we have

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{D} \otimes \Lambda^p \mathfrak{g}, \mathcal{D}) \cong \mathcal{D}^{\text{op}} \otimes \Lambda^p \mathfrak{g}^* \cong \mathcal{D}^{\text{op}} \otimes \Lambda^{m-p} \mathfrak{g} \otimes \Lambda^m \mathfrak{g}^*, \quad \forall p = 0, \dots$$

Thus, using resolution (5.2.3), we deduce that the object  $R\mathcal{H}om_{\mathcal{D}}(\mathcal{D}/\mathcal{D}\mathfrak{g}^\psi, \mathcal{D})$  may be represented by the complex

$$\mathcal{D}^{\text{op}} \otimes \Lambda^m \mathfrak{g} \otimes \Lambda^m \mathfrak{g}^* \rightarrow \mathcal{D}^{\text{op}} \otimes \Lambda^{m-1} \mathfrak{g} \otimes \Lambda^m \mathfrak{g}^* \rightarrow \dots \rightarrow \mathcal{D}^{\text{op}} \otimes \mathfrak{g} \otimes \Lambda^m \mathfrak{g}^* \rightarrow \mathcal{D}^{\text{op}} \otimes \Lambda^m \mathfrak{g}^*.$$

But this complex is acyclic in positive degrees since the corresponding associated graded complex is nothing but the resolution (5.2.4), tensored by  $\Lambda^m \mathfrak{g}^*$ . Further, the cokernel of the rightmost map  $\mathcal{D}^{\text{op}} \otimes \mathfrak{g} \otimes \Lambda^m \mathfrak{g}^* \rightarrow \mathcal{D}^{\text{op}} \otimes \Lambda^m \mathfrak{g}^*$ , in the above complex, is equal to  $\mathcal{D}^{\text{op}}/\mathcal{D}^{\text{op}}\mathfrak{g}^{\delta-\psi}$ . The result follows.  $\square$

**Corollary 6.1.2.** *For any  $\lambda, \nu \in \mathfrak{t}^*$  and  $c \in \mathbb{C}$ , there is a canonical isomorphism*

$$\mathbb{D}(\mathcal{V}_{\lambda,\nu,c}) \simeq \omega_{\mathcal{B} \times \mathcal{B} \times \mathbb{P}}^{-1} \otimes_{\mathcal{O}_{\mathcal{B} \times \mathcal{B} \times \mathbb{P}}} \mathcal{V}_{-\lambda-2\rho, -\nu-2\rho, -c-n}.$$

$\square$

**6.2. Duality for the Harish-Chandra  $\mathcal{D}$ -module.** Recall the automorphism  $\tau : \mathfrak{Z} \rightarrow \mathfrak{Z}$  induced by the isomorphism  $\tau : \mathcal{U}\mathfrak{g} \rightarrow (\mathcal{U}\mathfrak{g})^{\text{op}}$ , see §5.3.

**Proposition 6.2.1.** *For any  $(\theta, c) \in \text{Spec } \mathfrak{Z} \times \mathbb{C}$ , there is a canonical isomorphism*

$$\mathbb{D}(\mathcal{G}^{\theta, c}) \simeq \omega_{\mathbb{P}}^{-1} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{G}^{\tau(\theta), -c-n};$$

*Proof.* We choose and fix a finite dimensional vector subspace  $\mathfrak{H}_{\theta} \subset \mathfrak{Z}_{\theta}$  that freely generates the center, i.e., such the the imbedding  $\mathfrak{H} \hookrightarrow \mathfrak{Z}_{\theta}$  induces an algebra isomorphism  $\text{Sym } \mathfrak{H}_{\theta} \xrightarrow{\sim} \mathfrak{Z}$ . We will view the vector space  $\mathfrak{H}_{\theta}$  as an abelian Lie subalgebra of the enveloping algebra  $\mathcal{U}\mathfrak{g}$ .

We put  $\mathfrak{G} := \mathfrak{g} \oplus \mathfrak{H}_{\theta}$ , and view this direct sum as a filtered Lie algebra, the direct sum of the Lie algebra  $\mathfrak{g}$ , placed in filtration degree 1, and  $\mathfrak{H}_{\theta}$ , and abelian Lie algebra equipped with filtration induced by the standard filtration on the enveloping algebra  $\mathcal{U}\mathfrak{g}$ .

We imbed  $\mathcal{U}\mathfrak{g} \hookrightarrow \mathcal{D}(G)$  as left-invariant differential operators. This imbedding restricts to a filtration preserving map  $\mathfrak{H}_{\theta} \rightarrow \mathcal{D}(G)$ . Combining this map with the natural map  $\mathfrak{g} \rightarrow \mathcal{D}(\mathfrak{X}, c)$ , induced by the  $G$ -action of the group  $G$  on itself by left translations, we get a filtration preserving Lie algebra map  $\mathfrak{G} = \mathfrak{g} \oplus \mathfrak{H}_{\theta} \rightarrow \mathcal{D}(\mathfrak{X}, c)$ .

*Claim 6.2.2.* The corresponding moment map  $\mu : T^*\mathfrak{X} \rightarrow (\text{gr } \mathfrak{G})^*$  is flat.

This claim is a reformulation of [GG], Proposition 2.5; the map denoted by  $\mu \times \pi : T^*\mathfrak{X} \rightarrow \mathfrak{g} \times \mathbb{C}^{(n)}$  in *loc cit* is nothing but the moment map  $\mu$ , of the claim above.

By Claim 6.2.2, we are in a position to apply Lemma 5.2.1. According to the latter, the complex

$$0 \rightarrow \mathcal{D}_{\mathfrak{X}}^c \otimes \Lambda^{\text{top}}(\mathfrak{g} \oplus \mathfrak{H}_{\theta}) \rightarrow \dots \rightarrow \mathcal{D}_{\mathfrak{X}}^c \otimes \Lambda^1(\mathfrak{g} \oplus \mathfrak{H}_{\theta}) \rightarrow \mathcal{D}_{\mathfrak{X}}^c \rightarrow 0, \quad (6.2.3)$$

provides a resolution of the left  $\mathcal{D}_{\mathfrak{X}}^c$ -module  $\mathcal{G}^{\theta, c} = \mathcal{D}_{\mathfrak{X}}^c / \mathcal{D}_{\mathfrak{X}}^c(\mathfrak{g} \oplus \mathfrak{H}_{\theta})$ .

We now complete the proof of Proposition 6.2.1(i). The algebra  $\Gamma(\mathbb{P}, \mathcal{D}_{\mathbb{P}}^c)$  of global sections is the quotient algebra  $\mathcal{U}_c$ , of  $\mathcal{U}\mathfrak{g}$ . The anti-involution  $\tau : \mathcal{U}\mathfrak{g} \rightarrow (\mathcal{U}\mathfrak{g})^{\text{op}}$  induces an isomorphism  $\tau : \mathcal{U}_c^{\text{op}} \simeq \mathcal{U}_{-n-c}$ . Recall that  $\omega_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-n)$ . The canonical isomorphism  $(\mathcal{D}_{\mathbb{P}})^{\text{op}} \simeq \mathcal{D}_{\mathbb{P}, -n-c}$  coincides with  $\tau$  at the level of global sections.

We choose a left-invariant nonvanishing top degree differential form  $\beta$  on  $G$ . This form is also right-invariant, and it trivializes the canonical bundle  $\omega_G$ .

The algebra  $\mathcal{D}(G)$  is isomorphic to the smash product  $\mathbb{C}[G] \ltimes \mathcal{U}\mathfrak{g}$  (we embed  $\mathcal{U}\mathfrak{g}$  into  $\mathcal{D}(G)$  as left-invariant differential operators). Using the trivialization of  $\omega_G$  we get a canonical isomorphism  $\mathcal{D}(G)^{\text{op}} \simeq \mathcal{D}(G)$ . For  $h \in \mathbb{C}[G]$ , and  $x \in \mathfrak{g} \subset \mathcal{U}\mathfrak{g}$ , the action of the anti-involution is described as  $h \otimes 1 \mapsto h \otimes 1$ ,  $1 \otimes x \mapsto -1 \otimes x$ . In particular, the anti-involution restricts to  $\tau$  on  $\mathcal{U}\mathfrak{g}$ . So we keep the name  $\tau$  for the anti-involution of  $\mathcal{D}(G)$ . More generally, for any vector field  $v$  on  $G$  we have  $\tau(v) = -v + \frac{\text{Lie}_v \beta}{\beta}$ , cf. [K2], 2.7.1. In particular, if  $v$  is a *right*-invariant vector field, we have  $\tau(v) = -v$ . Moreover, the adjoint action of  $G$  on itself gives rise to the embedding  $\text{ad} : \mathfrak{g} \hookrightarrow \mathcal{D}(G)$ , and we have  $\tau(\text{ad}(x)) = -\text{ad}(x) = \text{ad}(-x)$ .

To compute  $\mathbb{D}(\mathcal{G}^{\theta, c})$ , the dual of the Harish-Chandra module, we use the free  $\mathcal{D}_{\mathfrak{X}}^c$ -resolution (6.2.3). Choose a trivialization  $\Lambda^{\text{top}}(\mathfrak{g} \oplus \mathfrak{H}_{\theta}) \simeq \mathbb{C}$  of the one dimensional vector space  $\Lambda^{\text{top}}(\mathfrak{g} \oplus \mathfrak{H}_{\theta})$ . Hence we obtain a perfect pairing

$$\Lambda^k(\mathfrak{g} \oplus \mathfrak{H}_{\theta}) \times \Lambda^{n^2+n-2-k}(\mathfrak{g} \oplus \mathfrak{H}) \rightarrow \mathbb{C}.$$

We apply the functor  $\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}}^c}(-, \mathcal{D}_{\mathfrak{X}}^c)$  to the resolution (6.2.3). Thus, we see that the object  $R\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}}^c}(\mathcal{G}^{\theta, c}, \mathcal{D}_{\mathfrak{X}}^c)$  is represented by the following complex of *right*  $\mathcal{D}_{\mathfrak{X}}^c$ -modules

$$0 \rightarrow \Lambda^{\text{top}}(\mathfrak{g} \oplus \mathfrak{H}) \otimes \mathcal{D}_{\mathfrak{X}}^c \rightarrow \dots \rightarrow \Lambda^1(\mathfrak{g} \oplus \mathfrak{H}) \otimes \mathcal{D}_{\mathfrak{X}}^c \rightarrow \mathcal{D}_{\mathfrak{X}}^c$$

arising from the action of  $\mathfrak{g} \oplus \mathfrak{H} \subset \mathcal{D}_{\mathfrak{X}}^c$  on  $\mathcal{D}_{\mathfrak{X}}^c$  by the *left* multiplication. The above complex is acyclic everywhere except the rightmost term, by Claim 6.2.2 again.

Combining this description of the right  $\mathcal{D}_{\mathfrak{X}}^c$ -module  $\mathcal{E}xt_{\mathcal{D}_{\mathfrak{X}}^c}^{n^2+n-2}(\mathcal{G}^{\theta,c}, \mathcal{D}_{\mathfrak{X}}^c)$  with the above description of the isomorphism

$$\tau \boxtimes \tau : (\mathcal{D}_{\mathfrak{X}}^c)^{\text{op}} = \mathcal{D}_G^{\text{op}} \boxtimes (\mathcal{D}_{\mathbb{P}}^c)^{\text{op}} \simeq \mathcal{D}_G \boxtimes \mathcal{D}_{\mathbb{P}, -n-c} = \mathcal{D}_{\mathfrak{X}, -n-c}$$

we see that

$$(\tau \boxtimes \tau) \left( \mathcal{E}xt_{\mathcal{D}_{\mathfrak{X}}^c}^{n^2+n-2}(\mathcal{G}^{\theta,c}, \mathcal{D}_{\mathfrak{X}}^c) \right) \simeq \mathcal{G}^{\tau(\theta), -n-c}$$

We conclude that  $\mathbb{D}(\mathcal{G}^{\theta,c}) \simeq \mathcal{O}_{\mathbb{P}}(n) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{G}^{\tau(\theta), -n-c}$ . The proposition is proved.  $\square$

From Claim 6.2.2, using Lemma 5.2.1, we deduce

**Corollary 6.2.4.** *For any  $(\theta, c) \in \text{Spec } \mathfrak{Z} \times \mathbb{C}$ , the characteristic cycle  $[SS(\mathcal{G}^{\theta,c})]$  equals  $[\mu^{-1}(0)]$ , the cycle of the scheme-theoretic zero fiber of the moment map  $\mu : T^*\mathfrak{X} \rightarrow (\mathfrak{gr } \mathfrak{G})^*$ .  $\square$*

**6.3. Reminder on affine Hecke algebras.** Recall that  $\mathbb{T} = (\mathbb{C}^\times)^{n-1}$  stands for the abstract Cartan torus of the group  $SL_n$ . Let  $\check{\mathbb{T}}$  denote the dual torus, so that  $\text{Hom}(\check{\mathbb{T}}, \mathbb{C}^\times) = \text{Hom}(\mathbb{C}^\times, \mathbb{T})$  is a lattice in  $\mathfrak{t}$ . The symmetric group  $W = \mathbb{S}_n$  acts naturally on  $\mathbb{T}$ ,  $\check{\mathbb{T}}$ , hence also on  $\mathbb{C}[\check{\mathbb{T}}]$ , a Laurent polynomial ring. Let  $\mathbb{C}[\check{\mathbb{T}}/W] = \mathbb{C}[\check{\mathbb{T}}]^W \subset \mathbb{C}[\check{\mathbb{T}}]$  denote the subalgebra of  $W$ -invariants.

Associated with any  $\Theta \in \check{\mathbb{T}}/W$ , there is a maximal ideal in  $\mathbb{C}[\check{\mathbb{T}}]^W$ . We let  $\mathbb{I}_\Theta \subset \mathbb{C}[\check{\mathbb{T}}]$  denote the ideal generated by that maximal ideal of the subalgebra  $\mathbb{C}[\check{\mathbb{T}}]^W$ . The quotient,  $R_\Theta := \mathbb{C}[\check{\mathbb{T}}]/\mathbb{I}_\Theta$ , called ‘coinvariant algebra’, is a vector space of dimension  $n!$  that comes equipped with the regular representation of the group  $W$ .

Given a complex number  $q \in \mathbb{C}^\times$ , let  $\mathcal{H}_q$  be the affine Hecke algebra of type  $\mathbf{A}_{n-1}$ , modelled on  $W \ltimes \text{Hom}(\check{\mathbb{T}}, \mathbb{C}^\times)$ , an affine Weyl group. Thus, there is a standard commutative subalgebra  $\mathbb{C}[\check{\mathbb{T}}] \subset \mathcal{H}_q$ , the Bernstein subalgebra, such that the corresponding  $W$ -invariants,  $\mathbb{C}[\check{\mathbb{T}}]^W \subset \mathbb{C}[\check{\mathbb{T}}] \subset \mathcal{H}_q$ , form the center of the affine Hecke algebra. There is a natural  $\mathbb{C}[\check{\mathbb{T}}]^W$ -linear action of the algebra  $\mathcal{H}_q$  on  $\mathbb{C}[\check{\mathbb{T}}]$  via so-called Demazure-Lusztig operators, cf. eg. [CG], ch. 7. In particular, for any  $\Theta \in \check{\mathbb{T}}/W$ , the coinvariant algebra  $R_\Theta = \mathbb{C}[\check{\mathbb{T}}]/\mathbb{I}_\Theta$  inherits an  $\mathcal{H}_q$ -module structure.

Let  $\mathbb{T}^{\text{reg}} \subset \mathbb{T} = (\mathbb{C}^\times)^{n-1}$  be the complement of the big diagonal, the subset of points with pairwise distinct coordinates. Let  $\mathfrak{X}^{\text{reg}} \subset \mathfrak{X} = SL_n \times \mathbb{P}$ , be an open subset formed by the pairs  $(g, \ell)$ , such that  $g \in G$ , is a regular semisimple element, and such that the line  $\ell$  is *cyclic* for  $g$ , i.e., such that we have  $\mathbb{C}[g]\ell = \mathbb{C}^n$ . We write  $\text{spec} : \mathfrak{X}^{\text{reg}} \twoheadrightarrow \mathbb{T}^{\text{reg}}/W$ ,  $(g, \ell) \mapsto \text{Spec}(g)$ , for the map that assigns to a pair  $(g, \ell)$  the unordered  $n$ -tuple of eigenvalues of the matrix  $g$ .

The fundamental group of the space  $\mathbb{T}^{\text{reg}}/W$  is known to be the affine braid group  $B_n^{\text{aff}}$ . Thus, choosing a base point  $x \in \mathbb{T}^{\text{reg}}/W$ , for any  $q \in \mathbb{C}^\times$ , we have a diagram

$$a : \pi_1(\mathbb{T}^{\text{reg}}/W, x) = B_n^{\text{aff}} \longrightarrow \mathcal{H}_q, \quad (6.3.1)$$

Given a pair  $(\Theta, q) \in \check{\mathbb{T}}/W \times \mathbb{C}^\times$ , let  $a^*(R_\Theta)$  be the pull-back of the  $\mathcal{H}_q$ -module  $R_\Theta$  via the map (6.3.1). Associated with  $a^*(R_\Theta)$ , one has a local system on  $\mathbb{T}^{\text{reg}}/W$ , and we write  $\mathcal{R}_{\Theta,q}$  for the pull-back of the latter local system to  $\mathfrak{X}^{\text{reg}}$  via the map  $\text{spec}$ .

**6.4. Monodromy conjecture.** According to [FG], Proposition 3.2.3, there is a canonical rational  $G$ -invariant section  $\mathbf{f}$ , of the line bundle  $\omega_{\mathfrak{X}}^{\otimes 2}$ , such that  $\mathbf{f}$  has neither zeros nor poles on the open set  $\mathfrak{X}^{\text{reg}}$ . Observe further that, for any  $c \in \mathbb{C}$  and any differential operator  $u$  on  $\mathfrak{X}^{\text{reg}}$ , the formula  $\mathbf{f}^{-c} \circ u \circ \mathbf{f}^c$  gives a well defined twisted differential operator  $\mathbf{f}^{-c} \circ u \circ \mathbf{f}^c \in \mathcal{D}_{\mathfrak{X}}^{2nc}|_{\mathfrak{X}^{\text{reg}}}$ . It follows that the assignment  $u \mapsto \mathbf{f}^{-\frac{c}{2n}} \circ u \circ \mathbf{f}^{\frac{c}{2n}}$  induces an isomorphism  $\mathcal{D}_{\mathfrak{X}}|_{\mathfrak{X}^{\text{reg}}} \xrightarrow{\sim} \mathcal{D}_{\mathfrak{X}}^c|_{\mathfrak{X}^{\text{reg}}}$ , of TDO.

We conclude that, given a  $\mathcal{D}_{\mathfrak{X}}^c$ -module  $\mathcal{M}$ , one may view  $\mathcal{M}|_{\mathfrak{X}^{\text{reg}}}$ , the restriction of  $\mathcal{M}$  to the open set  $\mathfrak{X}^{\text{reg}}$ , as a  $\mathcal{D}_{\mathfrak{X}^{\text{reg}}}$ -module via the above isomorphism of TDO.

The map  $\mathfrak{t}^* \twoheadrightarrow \mathfrak{t}^*/\mathfrak{t}_{\mathbb{Z}}^* = \check{\mathbb{T}}$  induces a canonical projection  $\mathfrak{t}^*/W \twoheadrightarrow \check{\mathbb{T}}/W$ .

We call a complex number  $c \in \mathbb{C}$  ‘good’ if it is not a negative rational number of the form  $c = -p/m$  where  $2 \leq m \leq n$ ,  $1 \leq p \leq m$ , and  $(p, m) = 1$ .

**Conjecture 6.4.1.** *Let  $c \in \mathbb{C}$  be good. We put  $q := \exp(2\pi ic)$ , and let  $\Theta \in \check{\mathbb{T}}/W$  be the image of an element  $\theta \in \mathfrak{t}^*/W$  under the canonical projection.*

*Then, the locally constant sheaf on  $\mathfrak{X}^{\text{reg}}$  associated with the  $\mathcal{D}$ -module  $\mathcal{G}^{\theta, c}|_{\mathfrak{X}^{\text{reg}}}$  via the Riemann-Hilbert correspondence is isomorphic to  $\mathcal{R}_{\Theta, q}$ .*

Let  $H_\kappa$  be the trigonometric Cherednik algebra with parameter  $\kappa = c/n$ , and let  $eH_\kappa e$  denote the corresponding spherical subalgebra, cf. [FG, §5]. The condition that  $c$  be good insures, by [BE], that the algebras  $H_\kappa$  and  $eH_\kappa e$  are Morita equivalent, cf. [FG, Proposition 3.1.3].

Recall further that, according to [GG] and [FG], the space  $\mathbb{H}(\mathcal{G}^{\theta, c}) := \Gamma(\mathfrak{X}, \mathcal{G}^{\theta, c})^{\text{sl}_n(\mathbb{C})}$ , called the *Hamiltonian reduction* of the  $\mathcal{D}$ -module  $\mathcal{G}^{\theta, c}$ , has a natural  $eH_\kappa e$ -module structure. It is clear that describing the de Rham local system of the  $\mathcal{D}$ -module  $\mathcal{G}^{\lambda, c}|_{\mathfrak{X}^{\text{reg}}}$  amounts to studying the monodromy of the corresponding  $\mathcal{D}$ -module  $\mathcal{D}(\mathbb{T}^{\text{reg}}/W) \otimes_{eH_\kappa e} \mathbb{H}(\mathcal{G}^{\theta, c})$ , on  $\mathbb{T}^{\text{reg}}/W$ .

Assume that  $c$  is good, so the above mentioned Morita equivalence holds. Then, the latter problem is equivalent to a similar problem for the  $H_\kappa$ -module,  $\mathcal{P}_{\theta, c}$ , that corresponds to the  $eH_\kappa e$ -module  $\mathbb{H}(\mathcal{G}^{\theta, c})$  via the Morita equivalence.

The  $H_\kappa$ -module  $\mathcal{P}_{\theta, c}$  has been computed in [GG], Lemma 7.5 in the rational case. In the trigonometric setting of the present section, the corresponding result reads

$$\mathcal{P}_{\theta, c} = H_\kappa \otimes_{\mathbb{C}[\check{\mathbb{T}}] \rtimes W} R_\Theta, \quad (6.4.2)$$

is the  $H_\kappa$ -module induced from the representation  $R_\Theta$ , of the subalgebra  $\mathbb{C}[\check{\mathbb{T}}] \rtimes W$ . The corresponding local system on  $\mathbb{T}^{\text{reg}}/W$ , comes from an  $W$ -equivariant  $\mathcal{D}$ -module,  $\text{KZ}(\mathcal{P}_{\theta, c})$ , on  $\mathbb{T}^{\text{reg}}$ , cf. [GGOR] for details about the functor KZ. The latter  $\mathcal{D}$ -module is nothing but the Knizhnik-Zamolodchikov connection in the trivial vector bundle on  $\mathbb{T}^{\text{reg}}$  associated with the representation  $R_\Theta$ ; the formula for the connection can be found e.g. in [FV], §3.1.

Thus, we conclude that our original problem about the monodromy of the Harish-Chandra  $\mathcal{D}$ -module  $\mathcal{G}^{\theta, c}|_{\mathfrak{X}^{\text{reg}}}$  is equivalent to a similar problem for the Knizhnik-Zamolodchikov connection in the representation  $R_\Theta$ .

For general enough  $c \in \mathbb{C}$ , the monodromy of the Knizhnik-Zamolodchikov connection that arises from (6.4.2) has been studied in [Ch1], Theorem 3.3; [Ch2], Theorem 3.6, and also [Op], Corollary 6.9. The results in *loc cit* confirm that our Conjecture 6.4.1 does hold for sufficiently general parameters  $c \in \mathbb{C}$ .

*Remark 6.4.3.* Let  $\mathcal{P}'_c = H_\kappa \otimes_{\mathbb{C}[\check{\mathbb{T}}] \rtimes W} \mathbb{C}[W]$ , be an  $H_\kappa$ -module induced from the regular representation of the group  $W$ , equipped with the ‘trivial’  $\mathbb{C}[\check{\mathbb{T}}]$ -action, via the morphism  $\mathbb{C}[\check{\mathbb{T}}] \rightarrow \mathbb{C}$ ,  $P \mapsto P(1)$ .

We note that an analogue of our monodromy conjecture, with the  $H_\kappa$ -module in (6.4.2) being replaced by the  $H_\kappa$ -module  $\mathcal{P}'_c$ , is known to be *false*, in general.

## REFERENCES

- [BB1] A. Beilinson, J. Bernstein, *Localisation de  $g$ -modules*. C. R. Acad. Sci. Paris Sr. I Math. 292 (1981), 15–18.
- [BB2] ———, ———, *A generalization of Casselman’s submodule theorem*, Progress in Math. **40** (1983), 35–52.
- [BB3] ———, ———, *A proof of Jantzen conjectures*. I. M. Gelfand Seminar, 1–50, Adv. Soviet Math., **16**, Part 1, Amer. Math. Soc., Providence, RI, 1993.
- [BG] ———, V. Ginzburg, *Wall-crossing functors and  $\mathcal{D}$ -modules*. Represent. Theory **3** (1999), 1–31.
- [BE] R. Bezrukavnikov, P. Etingof, *Induction and restriction functors for rational Cherednik algebras*. Selecta Math. (N.S.) **14** (2009), 397–425.
- [BFO] R. Bezrukavnikov, M. Finkelberg, and V. Ostrik, *Character  $D$ -modules via Drinfeld center of Harish-Chandra bimodules*. [arXiv:0902.1493](https://arxiv.org/abs/0902.1493).
- [Bo] A. Borel, *Algebraic  $D$ -modules*. Perspectives in Mathematics, 2. Academic Press, Inc., Boston, MA, 1987.

- [BoBr] W. Borho, J.-L. Brylinski, *Differential operators on homogeneous spaces*. II. Relative enveloping algebras. Bull. Soc. Math. France **117** (1989), 167–210.
- [Ch1] I. Cherednik, *Integration of quantum many-body problems by affine Knizhnik-Zamolodchikov equations*. Adv. Math. **106** (1994), 65–95.
- [Ch2] ———, *Lectures on Knizhnik-Zamolodchikov equations and Hecke algebras*. Quantum many-body problems and representation theory, 1–96, MSJ Mem., 1, Math. Soc. Japan, Tokyo, 1998.
- [CG] N. Chriss, V. Ginzburg, *Representation theory and complex geometry*. Birkhäuser Boston, 1997.
- [FV] G. Felder, A. Veselov, *Action of Coxeter groups on  $m$ -harmonic polynomials and KZ equations*. Mosc. Math. J. **3** (2003), 1269–1291. [arXiv:math.QA/0108012](#).
- [FG] M. Finkelberg, V. Ginzburg, *Cherednik algebras for algebraic curves*. [arXiv:0704.3494](#), 2007.
- [GG] W. L. Gan, V. Ginzburg, *Almost-commuting variety,  $D$ -modules, and Cherednik algebras*, IMRP **2006:2**, 1.
- [GK] I. Gelfand, D. Kazhdan, *Representations of the group  $GL(n, K)$  where  $K$  is a local field*. Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), pp. 95–118. Halsted, New York, 1975.
- [G1] V. Ginzburg,  *$\mathfrak{G}$ -modules, Springer’s representations and bivariant Chern classes*. Adv. in Math. **61** (1986), 1.
- [G2] ———, *Admissible modules on a symmetric space*, Astérisque **173-174** (1989), 199–256.
- [GGOR] ———, N. Guay, E. Opdam, R. Rouquier, *On the category  $\mathcal{O}$  for rational Cherednik algebras*. Invent. Math. **154** (2003), 617–651.
- [H] M. P. Holland, *Quantization of the Marsden-Weinstein reduction for extended Dynkin quivers*, Ann. Sci. École Norm. Sup., **32** (1999), 813–834.
- [HK1] R. Hotta, M. Kashiwara, *The invariant holonomic system on a semisimple Lie algebra*. Invent. Math. **75** (1984), 327–358.
- [HK2] ———, ———, *Quotients of the Harish-Chandra system by primitive ideals*, Progr. Math. **60** (1985), 185–205.
- [HTT] R. Hotta, K. Takeuchi, T. Tanisaki,  *$D$ -modules, perverse sheaves, and representation theory*. Progress in Mathematics, 236. Birkhäuser Boston, Inc., Boston, MA, 2008.
- [K1] M. Kashiwara,  *$B$ -functions and holonomic systems. Rationality of roots of  $B$ -functions*. Invent. Math. **38** (1976/77), 33–53.
- [K2] ———, *Representation theory and  $D$ -modules on flag varieties*, Astérisque **173-174** (1989), 55–110.
- [K3] ———, *The universal Verma module and the  $b$ -function*. Algebraic groups and related topics (Kyoto/Nagoya, 1983), 67–81, Adv. Stud. Pure Math., 6, North-Holland, Amsterdam, 1985.
- [KKV] F. Knop, H. Kraft, and T. Vust, *The Picard group of a  $G$ -variety*. Algebraische Transformationsgruppen und Invariantentheorie, 77–87, DMV Sem., **13**, Birkhäuser, Basel, 1989.
- [Lu] G. Lusztig, *Character sheaves I*. Adv. in Math. **56** (1985), 193–237.
- [MV] I. Mirković, K. Vilonen, *Characteristic varieties of character sheaves*, Invent. Math. **93** (1988), 405–418.
- [MWZ] P. Magyar, J. Weyman, A. Zelevinsky, *Multiple Flag Varieties of Finite Type*, Adv. Math. **141** (1999), 97–118.
- [Op] E. M. Opdam, *Lecture Notes on Dunkl Operators for Real and Complex Reflection Groups*, Math. Soc. of Japan Memoirs **8** (2000).
- [Sc] G. Schwarz, *Lifting differential operators from orbit spaces*. Ann. Sci. École Norm. Sup. **28** (1995), 253–305.
- [T] R. Travkin, *Mirabolic Robinson-Shensted-Knuth correspondence*, Selecta Mathem. **14** (2009), 727–758.

- **M.F.:** IMU, IITP, and State University Higher School of Economics, Dept. of Mathematics, 20 Myasnitskaya st. Moscow 101000 Russia;  
[fnklberg@gmail.com](mailto:fnklberg@gmail.com)
- **V.G.:** Department of Mathematics, University of Chicago, Chicago IL 60637, USA;  
[ginzburg@math.uchicago.edu](mailto:ginzburg@math.uchicago.edu)